ELEMENTARY SOLID GEOMETRY

ELEMENTARY SOLID GEOMETRY

INCLUDING THE MENSURATION OF THE SIMPLER SOLIDS

WITH 400 EXAMPLES

BY

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PREFACE

I HAVE much pleasure in acceding to the author's request that I should say a few words by way of preface to his book, for the main lines on which it is constructed have my cordial assent, and have indeed been followed, more or less, in the teaching given in this University for many years past.

The subject falls naturally into two main divisions. The first is a mere continuation of the Euclidean Geometry contained in Books I.–VI. of the Elements, and deals with the relations of straight lines, planes, and the simpler curved surfaces. There is no reason why this should not be treated as rigorously as the Euclidean plane geometry; and indeed nothing but vagueness and confusion could result from any attempt to deal with it otherwise. There is, however, one serious defect in Euclid's own exposition of the matter, in that various purely descriptive theorems relating to parallelism of lines and planes are made to depend on metrical properties involving perpendiculars, with which they have no natural connection. The author, deviating in this respect from the practice of Euclid and of most English text-books, has endeavoured to follow the strict scientific procedure. At the same time an attempt has been made to keep the number of cardinal propositions within moderate limits. As several otherwise excellent continental treatises show, this is not altogether an easy matter.

The second main division deals with the Mensuration of Solids and of Curved Surfaces. Here, I think, some greater latitude, and even laxity, of method is permissible. The subtleties involved in a rigorous treatment are best reserved for a later stage, when they are dealt with systematically in the Integral Calculus. In the case of elementary students, all that can fairly be required, and indeed all that is really practicable, is that simple "intuitional" proofs, or rather outlines of proofs, should be given, with an occasional warning (for conscience' sake) whenever an essential point is (for the present purpose) deliberately assumed without full demonstration.

Sections relating to the theory of Perspective, the theory of Symmetrical Figures, Euler's theorems on Polyhedra, and so on, have been interpolated. The ideas involved are simple, and the results are interesting; the inclusion of these topics is, moreover, amply warranted by precedent. In the present exposition special attention is directed to some points which are not infrequently overlooked.

HORACE LAMB.

VICTORIA UNIVERSITY OF MANCHESTER,
December 1906.

AUTHOR'S PREFACE

THE present book is designed for use in the Higher Forms of Schools and in University Courses of Intermediate Standard. Keeping both these ends in view, the number and range of Examples have been largely increased, and some of the more difficult portions of the book-work have been inserted at the ends of the chapters as Supplements.

The earlier figures are intended to suggest figures drawn on two blackboards placed at an angle to one another, in the hope that it will encourage the construction of actual solid figures, and that this will be found to help in bridging the gulf between the mental image of a solid and its conventional representation.

The Supplement which occurs at the end of the first part of the book, and which deals with Spherical Geometry, has been cast in its present form because of the light which is thereby thrown on the corresponding treatment of Plane Geometry, and because of its intrinsic interest as the Geometry of figures drawn on the Earth.

In any work which is the result of a gradual growth any acknowledgment of the sources from which help has been derived is necessarily incomplete. In the present case, I owe much to the Treatise on Geometry by Rouché and De Comberousse, from which book many examples have been taken; in Spherical Geometry similar help has been gained from the book on Spherical Trigonometry by Todhunter and

Lathem; and for references use has been made of the Short History of Mathematics, by W. Rouse Ball.

Much more do I owe to the opportunities of talking things over with Professor Lamb (who has also furnished a number of examples) and with my colleagues, Mr. R. F. Gwyther and Mr. F. T. Swanwick.

W. H. J.

VICTORIA UNIVERSITY OF MANCHESTER,
January 1907.

PREFACE TO THE SECOND EDITION

THE present edition differs from the first in that it contains additional matter, consisting of *Notes to Teachers* (p. xiii), giving the reasons for most of the differences between this book and other books covering the same ground; *References to Theorems in Plane Geometry* (p. 172), added because the Euclidean references adopted in the text are no longer so well known as formerly; and an *Alternative Treatment of the Fundamental Propositions* (p. 157), inserted because experience has shown that most pupils find more difficulty with the first chapter than with any other. As the natural result of six years' use, a number of alterations and corrections have also been made throughout the text.

I have again to thank my former colleague, Mr. F. T. Swanwick, as well as his successor, Mr. H. R. Hassé, for their help in talking over the alternative treatment now added and in supplying me with corrections for the text.

I shall be grateful for any hints which teachers may send me as a result of their experience in using this book.

W. H. J.

MANCHESTER, 1913.

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NOTES TO TEACHERS

"It is the glory of geometry that from those few principles, fetched from without, it is able to produce so many things. . . ."

"Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates of the art of measuring."

—From the Preface to Newton's *Principia*, 1686.

THE above quotations indicate the two general aims which the author has in mind: to provide a training in general method rather than a knowledge of particular theorems, and, by means of suitable examples, to suggest the relation between the exact abstract theory and its approximate concrete applications. In the first part care has been taken to help students to realise that it is no more possible for a statement to be *proved* unless in some form or other, tacitly or explicitly, it has already been *assumed*, than to create matter from a vacuum. Similarly, in the second part, dealing with mensuration, proofs have been chosen with the view of forming a *basis* for the integral calculus rather than of avoiding it wherever possible.

The style of proof has been purposely made rather curt, in the hope that the main points would thus therefore stand out more clearly, and that the effort necessary to grasp successive steps would fashion stouter pegs for the memory than are provided by the really deceptive lucidity of some excellent French text-books, for example. References to previous propositions are always accompanied by the page, in order to encourage frequent reference to the original proof and its accompanying figure. But it must not be supposed that this style is to be recommended for use by students in examination, where, in the absence of any commonly accepted reference numbers, any enunciation that may be used must be clearly indicated in its

entirety. That students are not given any encouragement to reproduce mechanically the wording of the text-book is not, by any means, necessarily a disadvantage. In the alternative treatment of Chapter I now given, an attempt is made to take both points of view into consideration, and the following plan has been adopted. Wherever possible, the enunciation has letters assigned to all the variables of which it treats, so that it may be used in subsequent proofs, word for word, without any change other than the substitution of the appropriate letters for those in the original enunciation.

The omission of problems or constructions is another point that requires explanation. In plane geometry constructions have a place of their own, because they correspond to actual operations performed by the student with ruler and compass. In solid geometry this is no longer the case, and they are reduced to their merely logical significance as existence theorems. This rôle may be best realised by a reference to the proofs of Proposition 12 (*a*) and (*b*), p. 168.

It is always difficult to find time to teach the student all the theorems he ought to know and at the same time give him practice in making simple proofs of his own. On this account a number of theorems, too numerous to rank as propositions and sufficiently simple to be set as examples, have been placed in the text and classed as scholiums.

ELEMENTARY SOLID GEOMETRY

PART I

PURE GEOMETRY

CHAPTER I

NON-METRICAL THEOREMS

1. Introductory.

The idea of a **solid body** is the simplest idea connected with space with which our experience makes us familiar. A **solid body** always possesses a boundary marking off the space we call inside the body from the space we call outside the body; this boundary is termed its **surface**.

We may think of the **surface** of a solid as consisting of two or more portions, which are separated by boundaries called **lines**.

Lastly, we may think of **lines** as consisting of portions which are separated by boundaries called **points**.

Solid Geometry aims at comparing solids in respect of shape, size, and position. These qualities are capable of being studied apart from the material bodies to which they originally belong; hence solid geometry is concerned solely with the position of points, lines, and surfaces, apart from all else.

Naturally, in the first place Geometry deals with the properties of the **simplest** surfaces and lines. Governing the mutual relations of planes and straight lines there are many laws, all of which are equally well based on experience. It

is the aim of Deductive Geometry to classify these laws, and, further, to select a few, called Postulates, which have the widest application, and then to show that all others are included in these.

Our concern is primarily, therefore, not with the truth or falsehood of the laws with which we deal, but rather with the relation in which they stand to one another; their truth, that is, their applicability to experience, can only be determined by trial.

It frequently happens that propositions are proved (that is, their truth is shown to depend on that of the postulates on which geometrical theory is based) by a method generally known as a *Reductio ad Absurdum.* A more accurate description of the method is obtained by referring to it as the **Method of Exhaustion.*** It may be stated as follows:—

If one of two or more mutually exclusive hypotheses is necessarily true, any one is proved true when all the rest have been proved not true.

For example, let a, b denote those sides of a triangle ABC opposite to the angles at A, B respectively.

Euc. i. 18 states that—

"When $a > b$, $A > B$," and "When $a < b$, $A < B$."

Taking this for granted, we can now see that Euc. i. 6, which states that—

"When $A = B$, $a = b$,"

is inevitably included.

For if a were greater or less than b, then A would be greater or less than B, and could therefore never be equal to B; that is, a cannot be greater or less than b, and must therefore be equal to it.

* Not to be confused with the Method of Exhaustions, used by Eudoxus (408-355 B.C.) in propositions such as Prop. 6, p. 126, which depend on the division of the figure into infinitesimal portions.

To take a more general example of the Method of Exhaustion, the proposition "When A is true, B is true" may also be expressed "When B is not true, A is not true," if we know that each of the statements denoted by A and B must either be true or not true. For every conceivable case must fall under one of the four classes in the following scheme:—

	B is true	B is not true
A is true	I	II
A is not true	III	IV

It is here meant that Class I. includes all cases in which A, B are both true; Class II. includes all cases in which A is true but B is not true; Class III., those in which A is not true and B is true; Class IV., those in which neither A nor B is true.

The proposition "When A is true, B is true," is equivalent to saying that Class II. does not exist. But the proposition "When B is not true, A is not true," is also equivalent to saying that Class II. does not exist. The two propositions are therefore identical.

Two propositions which are of the forms "When A is true, B is true," and "When B is true, A is true," are **converse** to each other. The **hypothesis** and **conclusion** of a proposition are interchanged in order to obtain its converse.

A proposition and its converse are not equivalent. This may be seen by reference to the above scheme.

The proposition "When A is true, B is true," states that Class II. does not exist; but the proposition "When B is true, A is true," states that Class III. does not exist.

For example, the converse of "When two triangles are congruent, their areas are equal," is "When two triangles are equal in area they are congruent," and the converse of the proposition, Euc. i. 8, "When two triangles have three corresponding sides equal they have three corresponding angles equal," is "When two triangles have three corresponding angles equal, they have three corresponding sides equal." But neither of these converse propositions is true.

2. Postulates.*

† Postulates of the Straight Line.

Post. 1. A straight line is of indefinite extent.

Post. 2. There is one straight line, and only one, passing through any two given points.

Postulates of the Plane.

Post. 3. Through any given point there is one straight line, and only one, which does not cut a given straight line, and which is in the same plane with it.

Post. 4. The whole of a straight line passing through any two points of a plane surface lies in that plane.

Post. 5. There is at least one plane passing through any three given points.

Post. 6. If two planes have one point in common they have another also.

These postulates, along with the initial statements, render formal definitions of a straight line and plane unnecessary.

‡ **Scholium:** A plane is of indefinite extent.

* See Appendix.

† A complete analysis of the postulates of geometry is given by D. Hilbert, *The Foundations of Geometry* (Trans.), London, 1902, and A. N. Whitehead, *The Axioms of Projective Geometry*, Cambridge, 1906.

‡ Scholium (=note) is here restricted to denote a deduction which it is useful to remember, but which hardly ranks as a proposition.

3. The Relations between Two Planes.

* **Def. 1.** Two planes which have no common point are said to be **parallel**.

Def. 2. Two planes (or straight lines) which are not coincident, but which have a common point, are said to **cut**; all the points common to two planes form their **join**, which is also called the **trace** of one plane upon the other.

PROP. 1. **Through three points not all in one straight line only one plane can be drawn.**

For if two planes α, β have points A, B, C in common, they also have in common the whole of the straight lines AB, AC. [Post. 4, p. 4.

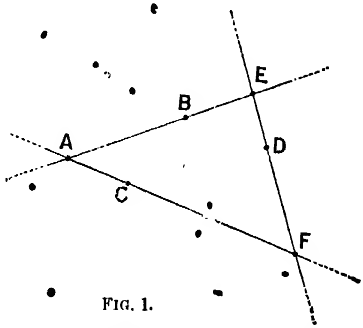


FIG. 1.

Let D be any point of the plane α. In the plane α, any straight line through D (with two possible exceptions) cuts both AB, AC; say at points E, F respectively. [Post. 3, p. 4.

These points E, F lie in the plane β, and therefore, so also does the point D. [Post. 4, p. 4.

That is, any point in α lies also in β, and the planes α, β are coincident.

There is therefore only one plane through the given points.

* In every case it is to be understood that a definition leaves quite untouched the question of the existence of the thing defined: that must ultimately depend on the postulates.

* Prop. 2. **If two planes have one point in common their join is a straight line.**

For, if they have one point in common, they have at least two; [Post. 6, p. 4.

And, if they have two points in common, they have in common the one straight line which contains both points. [Post. 2, Post. 4, p. 4.

Further, they can have in common no point outside this line, for then they would be coincident; [Prop. 1.

Their join is therefore a straight line.

The possible relations between two planes may now be stated:

Two planes must either
(i) be parallel,
or (ii) cut in a straight line.

Scholium: A plane is completely determined by either of the following data:
(i) three points not in one straight line,
(ii) a straight line and a point outside it,
(iii) two straight lines which cut.

4. The Relations between a Plane and a Straight Line.

Def. 1. A plane and a straight line which have no point in common are **parallel.**

A plane and a straight line may have in common either:
(i) no point, in which case they are parallel, [Def.
(ii) one point, when they cut,
or (iii) two or more points, when the plane contains the straight line. [Post. 4.

Def. 2. The point in which a straight line meets a plane is said to be its **trace** upon that plane.

* Euc. xi. 3.

5. The Relations between Two Straight Lines.

The plane through one straight line and one point of a second straight line must either

(i) cut it,
or (ii) contain it. [Post. 4, p. 4.

Since two planes which have in common a straight line and an external point are coincident (Sch., p. 6), it follows, if any one plane contains both of two given straight lines, that no plane which contains one straight line can cut the other in a point external to the first.

Also, if any one plane contains one line and cuts the other in a point external to the first, no plane can contain both straight lines (by the Method of Exhaustion).

Def. 1. Two straight lines such that some plane exists which contains both are said to be **co-planar**; two straight lines such that no plane exists which contains both are called **skew**.

Def. 2. Two co-planar straight lines which have no common point are **parallel**.

The possible relations between any two straight lines are shown in the following scheme:—

They may be

$$\left.\begin{array}{l}\text{intersecting} \\ \text{non-intersecting}\left\{\begin{array}{l}\text{parallel}\end{array}\right.\end{array}\right\}\text{co-planar.}$$
$$\qquad\qquad\qquad\qquad\quad\text{skew.}$$

Scholium (*a*): A straight line is parallel to, or contained by, a plane if it is parallel to any one straight line in that plane. [Defs.

Scholium (*b*): A plane is completely determined when it contains two parallel straight lines. [Post. 3, Def.

Scholium (*c*): A plane which contains one of two parallel straight lines, cannot cut the other.

6. Notation.

Points will be denoted by **capital letters**, such as A, B, C.

Straight lines by **small letters**, such as a, b, c.

Planes by small Greek letters, such as α, β, γ.

The **straight line** containing two **points**, A, B, will be denoted by AB, and the **plane** containing two **straight lines** a, b, by ab.

The **straight line** which is **common** to two **planes**, α, β, will be denoted by $(\alpha\beta)$.

7. Three Planes.

Prop. 3. **The joins of three planes no two of which are parallel are either concurrent or parallel; if two planes are parallel their joins with the third are parallel.***

For the three planes either

(i) have a point A common to all, [Fig. 2.

or (ii) have no point common to all. [Fig. 3.

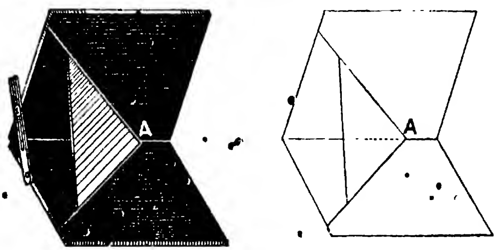


Fig. 2.

In the first case, A is common to every pair of planes, and the three joins meet at A.

In the second case, no two joins can cut (for that point would be common to all three planes) and, being co-planar in pairs, the joins are parallel. [Def. 2, p. 7.

* Euc. xi. 16.

Lastly, if two of the planes are parallel there can be no point common to the three planes, [Def. 1, p. 5.
and the two joins cannot cut; being co-planar they are therefore parallel. [Def. 2, p. 7.

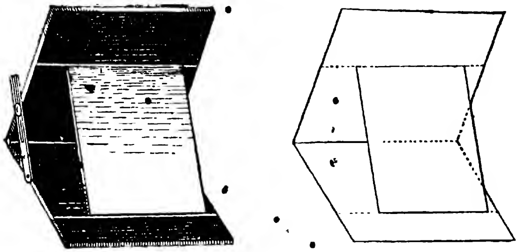


Fig. 3.

Prop. 4. **If two planes are each parallel to a third they are parallel to one another.**

If the two planes could have a common point A, their joins with any other plane, which passes through A and cuts the third plane in a line l, would be two lines parallel to l. [Prop. 3.

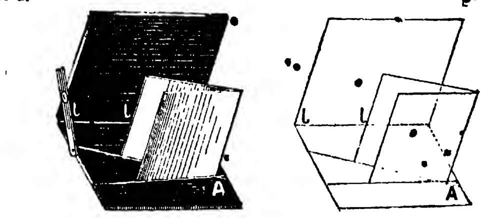


Fig. 4.

But both of these lines parallel to l would also pass through A, which is impossible. [Post. 3, p. 4.

The planes are therefore parallel.

Scholium: If a plane cuts one of two parallel planes it cuts the other also. [Cor. to Prop. 4.

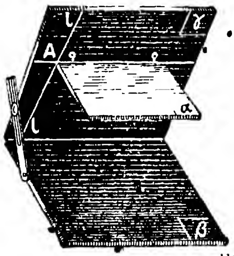
8. Two Planes and a Straight Line.

PROP. 5. **If two planes are parallel, a straight line which cuts one, cuts the other also.**

Consider a plane γ passing through the given line l and any point of the given plane β; and let l cut the given plane α in A.

γ having a point in common with each, cuts α, β, in parallel lines. [Props. 2, 3.

The line l cuts one of these at A and, lying in the same plane γ, must therefore cut the other also. [Post. 3, p. 4.



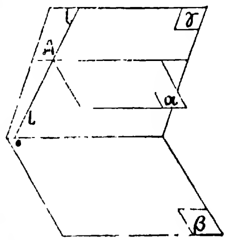


FIG. 5.

That is, if the given line cuts one plane it also cuts the other.

Scholium (a): A straight line which is parallel to, or contained by, one of two parallel planes is either parallel to or contained by the other. [Method of Exhaustion.

Scholium (b): If a straight line is contained by one plane and is parallel to another plane, it is parallel to their join. [Defs.

Scholium (c): A straight line which is parallel to each of two planes is parallel to their join.

(Consider a plane which contains the given line and any point of the join of the two planes and use Sch. (b).)

9. One Plane and Two Straight Lines.

Prop. 6. **Any plane which cuts one of two parallel straight lines cuts the other also**

The given plane *a* cuts the plane containing the parallel lines l, m, in a straight line which cuts l at some point A. [Prop. 2, p. 6.

This line must therefore cut m also. That is, if the given plane cuts one of the parallel lines it cuts the other also. [Post. 3, p. 4.

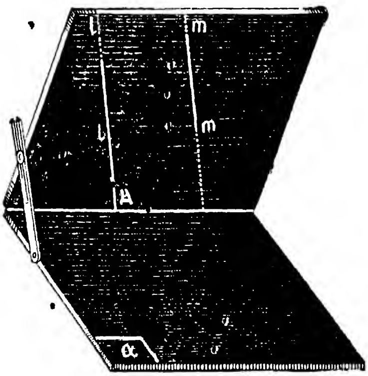


Fig. 6.

Scholium (*a*): If a plane is parallel to or contains one of two parallel straight lines it is parallel to or contains the other. [Method of Exhaustion.

Note that exactly the same statement may be made in different words as follows. If a straight line is parallel to or contained in a given plane, any straight line parallel to the given line is also either parallel to or contained in the given plane.

Scholium (*b*): If two intersecting lines are parallel to a given plane, the plane containing them is parallel to it. [Post. 3, p. 4.

***Scholium** (*c*): If two intersecting straight lines are respectively parallel to another pair of intersecting lines, the plane of the first pair cannot cut that of the second pair. [Prop. 3, Post. 3.

* Euc. xi. 15.

10. Three Straight Lines.

*Prop. 7. **If two straight lines are each parallel to a third they are parallel to one another.**

Let l, m be lines each parallel to n.

(i) l, m cannot cut. [Post. 3, p. 4.

(ii) They are co-planar; for otherwise, a plane containing l and one point of m, would cut m, would therefore cut n, [Prop. 6.

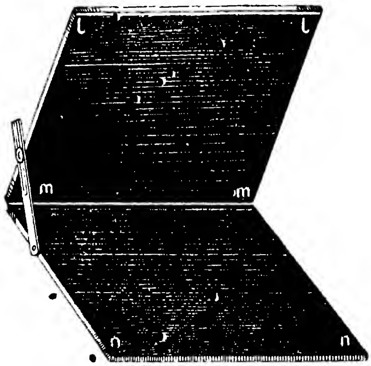


Fig. 7.

and therefore l also, [Prop. 6.
which it contains.

From (i) and (ii) it follows that l, m are parallel. [Def. 2, p. 7.

Scholium: Three straight lines, such that each cuts the other two, are concurrent or co-planar.

For any third straight line, which cuts each of two inter-

* Euc. xi. 9.

secting straight lines, must either (i) pass through their common point, or (ii) cut them in two distinct points. In this last case the lines are co-planar. [Post. 4, p. 4.

This reasoning can be applied equally well to any number of straight lines, such that each cuts all the others.

Examples I.

1. No two straight lines each joining two given skew straight lines are parallel to one another; no two are co-planar unless they cut on one of the given lines.

2. If any number of points are such that all sets of three are collinear they are all collinear.

3. If six points lie three by three on four straight lines, they are co-planar.

4. If three planes have no common point the join of any two is parallel to the third plane.

5. Through any point not common to two given planes an unlimited number of planes can be drawn, each cutting them in a pair of parallel straight lines.

6. There is one and only one plane containing one of two given skew straight lines and parallel to the other.

7. There is one and only one plane parallel to a given plane and containing a straight line parallel to it.

8. The locus of straight lines drawn through a given point parallel to a given plane is a parallel plane.

9. There is one and only one plane through a given point parallel to both of two skew straight lines.

10. Three planes having no common point must cut in two or three straight lines or not at all.

11. If a number of planes have two points common to all, any other plane cuts them in straight lines which are either all concurrent or all parallel.

12. Through any point, not lying in either of two parallel planes, one and only one straight line exists which cuts each of two given skew straight lines.

13. Parallel to any straight line (provided that this line is not parallel to a certain plane), there exists one and only one straight line which cuts each of two given skew straight lines.

14. Into how many portions, in general, do four planes taken at random divide space? [15.

15. Four planes, no two of which are parallel, meet in six straight lines which are such that either:

(i) each meets at least four others,
(ii) three are parallel.
(iii) six are parallel.

16. If n planes are such that no two are parallel, and that through no point do there pass more than three, they cut in $\frac{1}{2}n(n-1)$ straight lines; if, further, no two of these straight lines are parallel, they cut in $\frac{1}{6}n(n-1)(n-2)$ points.

17. A tetrahedron is uniquely determined when in each of its six edges one point is given; given six points there are thirty tetrahedrons, such that each edge contains one of the six points.

18.* There is an unlimited number of straight lines, mutually skew, which all cut two given skew straight lines, and which are all parallel to a given plane not parallel to the two given lines.

19.* There is an unlimited number of straight lines, mutually skew, which all cut three given straight lines, no two of which are co-planar.

SUPPLEMENT.

11. Perspective.

Def.† Two rectilinear figures are in **perspective** when to each point of one figure corresponds a point in the other

* If string models be constructed it will be found that these systems of lines lie on curved surfaces; they are respectively named paraboloids and hyperboloids. † See Def. 1, p. 17.

figure such that the line joining them passes through a fixed point called the centre of perspective.

Either figure may be said to be a conical projection of the other.

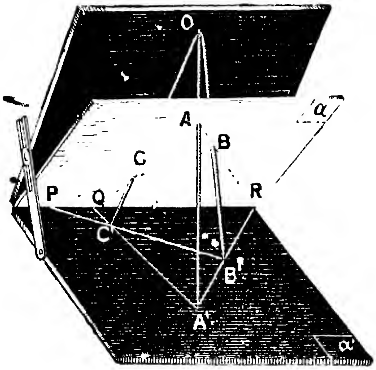


FIG. 8.

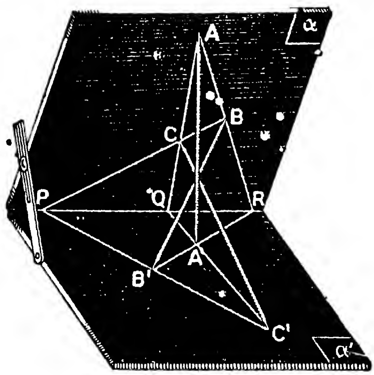


FIG. 9.

PROP. 8.* **If two plane figures not co-planar are in perspective corresponding straight lines either are parallel or intersect in collinear points.**

Let α, α' denote the planes of the two figures and AB, A′B′ two corresponding lines.

Then AA′, BB′ meet in the centre of perspective O and are therefore co-planar.

Hence AB, A′B′ both lying in the plane OAB, either cut one another or are parallel; and, if they cut, their common point R must be common to the planes α, α', and therefore lies on the straight line ($\alpha\alpha'$).

Thus all corresponding lines in these two planes either cut in collinear points or are parallel.

Def. A perspective drawing of an object is a plane figure in perspective with it; the centre of perspective being supposed placed in the eye of an observer, and the plane being called the Picture Plane (P.P.).

The following Rules can be immediately deduced from the preceding propositions:—

Rule (i). Parallel straight lines which are parallel to the Picture Plane are represented by parallel straight lines.

Rule (ii). Parallel straight lines which are not parallel to the Picture Plane are represented by concurrent straight lines meeting in a point called their Vanishing Point (V.P.); the line containing both this point and the Centre of Perspective is parallel to the given lines.

Rule (iii). The Vanishing Points of sets of parallel straight lines all parallel to a given plane lie in a straight line called its Vanishing Line (V.L.); the plane containing this line and the Centre of Perspective is parallel to the given plane.

* Desargues (1593–1662).

12. Projection on a Plane by Parallel Lines.

Def. 1. One figure is the **parallel projection** of the other when the lines joining corresponding points are all parallel. The figures are also said to be in **parallel perspective.**

Def. 2. The plane on which a figure is projected is called the **plane of projection.**

PROP. 9. **The parallel projection of a straight line on a plane is a straight line.**

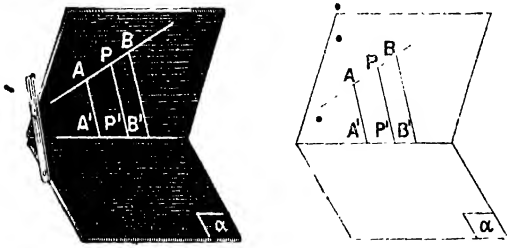


FIG. 10.

If A, B are any two points in a given straight line and A′ be the projection of A on a given plane α, P′ (the projection of P, any point on AB) lies in the plane ABA′ as well as in the plane α. [Def. 2, p. 7.

The projection of the straight line AB is therefore the straight line which is the join of these two planes. [Prop. 2, p. 6.

Scholium (*a*): The projections of parallel straight lines are parallel straight lines.

For if their projections had a common point the given lines, being co-planar, would have a corresponding point in common.

Scholium (*b*): If one plane figure is the projection of another, corresponding straight lines in the two figures either are parallel or cut in collinear points.

[*Cp.* Prop. 8.

13. Points at Infinity.*

In treating of the relations between two planes we said that they either

(i) cut in a straight line

or (ii) are parallel;

and in the theorems which followed, separate consideration had to be given to parallel planes. This double treatment can be avoided if we agree to speak of parallel planes as planes which "cut in a straight line at infinity."

It is useless to attempt to form any geometrical conception of "points at infinity," because it is simply a convenient form of speech by the use of which we can reduce all the theorems yet considered to special cases of a single more general theorem.

In the same way we may agree to speak of a straight line "cutting in a point at infinity" any plane or straight line to which it is parallel.

In order that these phrases may really simplify the relations which we wish to describe, it is important to see if it is possible to make these titular "points and straight lines at infinity" obey the six fundamental postulates (p. 4) in all or nearly all cases.

Def. 1. In the first place, we must define a straight line containing a point a, and X, a "point at infinity on the straight line a," to be a straight line containing P and parallel to a.

Def. 2. Similarly, a plane containing two points P, Q and X, a "point at infinity on the straight line a" is a plane containing P, Q and parallel to a.

Def. 3. Further, a plane containing a point P and X, Y, two "points at infinity on the straight lines a, b respectively," is a plane containing P and parallel to the straight lines a, b.

* The idea of Points at Infinity was used by Desargues in 1639.

The first two and the fourth postulates offer no difficulties, but the third must be expressed anew. It is equivalent to the following statement:—

In any given straight line there is one and only one "point at infinity," and every pair of co-planar straight lines have one point in common (which may be a "point at infinity").

For if a straight line a had two "points at infinity," X and Y, through any point P two straight lines (Post. 2), PX and PY, could be drawn each parallel to a (Def. 1, p. 18).

The fifth postulate, when one or two of the three points are taken to be "points at infinity," states the possibility of drawing a plane through two given points parallel to a given straight line (Def. 2, p. 18) or through one given point parallel to two given straight lines (Def. 3, p. 18).

The sixth postulate is only used to prove the second proposition, which enables us to make the following statement:—

Every pair of planes not coincident cut in a straight line, which may be a "straight line at infinity."

There is one exception to the new statement of the third postulate given above: A "straight line at infinity" contains an unlimited number of "points at infinity," instead of only one.

For a "straight line at infinity," x, implies the existence of two parallel planes, α and β say, which are such that an indefinitely great number of planes, such as γ, may be drawn, each plane cutting them in a pair of parallel straight lines, such as $(\alpha\gamma)$ and $(\beta\gamma)$, (Schol., p. 9). Every such pair of lines cut in a "point at infinity," and in order to preserve the same form of speech as before, these must be described as common to the two planes α, β, and therefore as contained by the "straight line at infinity," x.

Scholium: In any plane there is only one "line at infinity."

For if there were more, there would be more than one plane parallel to a given plane and passing through a given point.

Parallel projection as defined in § 12 is seen to be a special case of perspective if we take the centre of perspective to be a "point at infinity."

We can now express the results of this chapter as special cases of the following general theorem :—

Theorem: If X, Y, Z refer to any lines, surfaces, or spaces, then any points common to X and Y and to X and Z are common to X, Y, and Z, and therefore to Y and Z.

For example, to get the results of § 7, let X, Y, Z refer to three planes.

If the point common to the three planes is "at infinity," their joins have a "point at infinity" in common; in every case, therefore, the joins are concurrent or parallel.

[Prop. 3, p. 8.

If one of the joins becomes a "straight line at infinity," the other two "cut at infinity" and are parallel.

[Prop. 3, p. 8.

If two of the joins become "straight lines at infinity," they must coincide by the scholium above, and therefore all three joins coincide at infinity, and the three planes are each of them parallel to the other two. [Prop. 4, p. 9.

Examples II.

1. If two triangles are such that pairs of corresponding sides lie respectively in three distinct planes, the triangles are in perspective.

2. Two triangles, not co-planar, which have pairs of corresponding sides, either parallel or cutting in collinear points, are in perspective.

3. Any two triangles which have pairs of corresponding sides parallel or cutting in three collinear points are in perspective. (To be deduced from Ex. 2.)

4. If two tetrahedrons are in perspective, the six pairs of corresponding edges are parallel or cut in co-planar points, and the four pairs of corresponding planes are parallel or cut in co-planar straight lines.

Express the following statements (5–10) in terms of parallel straight lines and planes, instead of in terms of "points at infinity." Justify the original statement (in terms of "points at infinity") if necessary.

5. If two planes have a "straight line at infinity" in common, any straight line which cuts one plane "at infinity" cuts the other "at infinity" also, or is contained in it.

6. If two straight lines cut "at infinity," any plane which contains one or cuts it "at infinity" cuts the other "at infinity" also or contains it.

7. If a straight line cuts each of two planes "at infinity," it cuts their common join "at infinity."

8. If two intersecting straight lines each cut a given plane "at infinity," the plane containing them also cuts it "at infinity."

9. If two intersecting straight lines p, q respectively cut "at infinity" two other intersecting straight lines p′, q′, the planes pq, p′q′ cut "at infinity."

10. If two straight lines each cut "at infinity" a third straight line, they cut one another "at infinity."

11. If α, β, γ, δ be four planes which have no common point, even at infinity, the joins $\alpha\beta$ and $\gamma\delta$ are skew.

CHAPTER II

GENERAL METRICAL THEOREMS

14. The Angle between Two Straight Lines.

THE conceptions expressed in the postulates of the preceding chapter admit of remarkable developments, which form the subject of Projective Geometry; but at the same time, taken alone, they include but a small fraction of the facts connected with space which Geometry sets out to classify.

Other conceptions necessary to make a complete scheme* are those involved in the fact that it is possible to compare solid bodies, regarded as **rigid**, by bringing them together and placing them side by side.

The final postulate thus required is stated below in a form ready for immediate application.

Post. 7. A geometrical figure may be conceived to change its position in space without alteration of any of its geometrical properties; in particular, if α denote any plane of the figure, a any straight line in the plane, A any point in the straight line, and β, b, B denote any other plane, straight line, and point similarly related to one another, the figure can be moved so that α, a, A will become coincident with β, b, B respectively.

Scholium. The **position** of a geometrical figure is **fixed** when the positions of three of its points are known.

Def. Two geometrical figures are **congruent** when they can be made coincident by Superposition.

In plane geometry it is now usual to distinguish between **congruent** figures and equal figures, the latter being taken to mean that the figures are of equal area, but not necessarily of the same shape. At

* A. N. Whitehead, *The Axioms of Descriptive Geometry*. Cambridge. 1907.

the same time the word equal is not explicitly defined, but is left to be understood from the following axioms:—

Things which are equal to the same thing are equal to one another.
If equals be added to equals, the wholes are equal.
If equals be taken from equals, the remainders are equal.
Things which are halves of the same thing, or of equal things, are equal to one another.

A more concise method is indicated by the observation that the various geometrical magnitudes described as equal, lengths, angles, areas, can be completely specified by a single numerical measure.

Def. Geometrical magnitudes are equal when their numerical measures in terms of the same unit are the same.

In the case of lengths of straight lines, and magnitudes of angles, equality denotes congruence.

Equality is denoted by the sign " = ."

* Prop. 10. **If each of two intersecting straight lines is parallel to one of two other intersecting straight lines, the angles made by the first pair are respectively congruent with those made by the second pair.** (Fig. 11, p. 24.)

Let AB, AC be the first pair of lines, and let the second pair DE, DF be parallel to AB, AC respectively, and in the same direction.†

Then are the angles BAC, EDF to be proved equal.
Choose E, F so that DE = AB, DF = AC.

Because AB, DE are equal, parallel, and in the same direction, ABED is a parallelogram, [Euc. i. 33.

And therefore AD, BE are equal, parallel, and in the same direction. [Euc. i. 34.

Similarly AD, CF are equal, parallel, and in the same direction,

And therefore BE, CF are equal, parallel, and in the same direction. [Prop. 7, p. 12.

Hence BC, EF are equal; [Euc. i. 33, 34.

Therefore the triangles ABC, DEF are congruent; [Euc. i. 8.

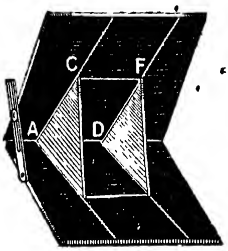
That is, the angles BAC, EDF are equal.

* Euc. xi. 10.
† Parallel straight lines AB, DE are drawn in the same direction when they lie on the same side of AD.

The proposition just proved enables us to give the following definition of the angle between two skew straight lines:—

Def. 1. The **angle** between two skew straight lines is the angle between two intersecting straight lines to which they are respectively parallel.

It is sometimes convenient to speak of parallel straight lines as all having the **same** direction, because this is a reminder that the angle between any two straight lines is equal to the angle between any pair of lines to which they are respectively parallel.



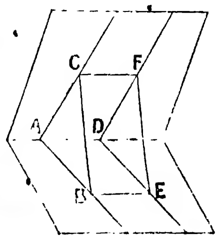


FIG. 11.

Def. 2. Two skew straight lines are perpendicular when the angles between them are right angles.

15. The Normal to a Plane.

Def. 3. A straight line is **normal** to a plane when it is perpendicular to all lines in that plane; any straight line which cuts a plane and is not normal to it is **oblique** to it.

* PROP. 11. **A straight line is normal to a plane if it is perpendicular to each of two intersecting straight lines in that plane.**

Let the line DG meet the given plane a in A.

* Euc. xi. 4.

Let AB, AC be straight lines parallel to the two given lines; they are therefore perpendicular to DG.
[Prop. 10, p. 23.

It is to be proved that DG is perpendicular to any other straight line in a.

Let AE be parallel to any other straight line in the plane, and let some straight line through E cut AB, AC in B, C respectively. Choose G so that DA = AG.

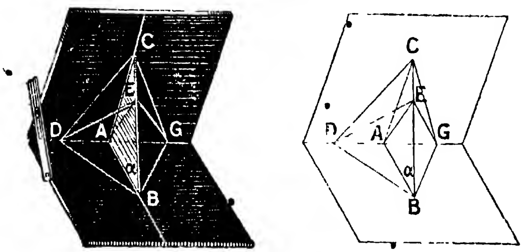


FIG. 12.

The triangles DAB, GAB are congruent, [Euc. i 4.
and therefore BD = BG.

Similarly, CD = CG. [Euc. i. 4.

Therefore the triangles BCD, BCG are congruent,
[Euc. i. 8.
and therefore also ED = EG by superposition.

Hence the triangles DAE, GAE are congruent, [Euc. i. 8.
and therefore DAE = GAE.

Hence DG is perpendicular to AE and to the line to which AE is parallel. That is, DG is perpendicular to any straight line in a and is therefore normal to a.

* PROP. 12. **Through any point there is only one plane normal to a given straight line, and only one straight line normal to a given plane.**

(i) For suppose that two planes through a point A were to cut a given straight line to which they were both normal in points B, B′.

The straight lines AB, AB′ would each be perpendicular to the given line BB′, and this is impossible. [Def. 3, p. 24; Euc. i. 16.

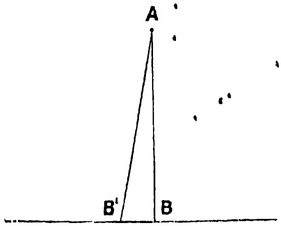


FIG. 13.

(ii) Again, suppose two straight lines through A were to cut a given plane to which they were both normal in points B, B′.

As before, AB, AB′ would each be perpendicular to BB′; which is impossible. [Def. 3, p. 24; Euc. i. 16.

Hence through a given point there is only one plane normal to a given straight line, and only one straight line normal to a given plane.

Scholium (*a*): The shortest distance of a point from a given plane is along the normal to the plane from the point. [Def. 3, p. 24; Euc. i. 16, 19.

Def. The **distance** of a point from a plane means the shortest distance of the point from the plane.

Scholium (*b*): The locus of points equidistant from two given points is the plane which bisects the straight line joining them and is normal to it. [Euc. i. 8; Prop. 12.

† **Scholium** (*c*): Planes normal to the same straight line are parallel. [Prop. 12, and Method of Exhaustion.

* Includes Euc. xi. 13. † Euc. xi. 14.

*PROP. 13. **Straight lines normal to the same plane are parallel.**

Let a, b be two normals to a plane α.

If through A, any point in a, a straight line be drawn parallel to b, it also will be normal to the plane.

[Def. 3, p. 24; Prop. 10, p. 23.

But through any point A there is only one line a normal to α. [Prop. 12.

Therefore the parallel to b through A is a; that is, a and b are parallel.

Scholium (a): A straight line perpendicular to each of two non-parallel straight lines is in a definite direction.

[Prop. 11, p. 24; Prop. 13.

Scholium (b): A straight line perpendicular to a given straight line is not fixed in direction, but is parallel to all planes normal to the given line. [Method of Exhaustion.

Def. 1. The **orthogonal projection** of a point on a plane is the point in which the normal through it cuts the plane.

Scholium (c): The orthogonal projection of a straight line is a straight line or a point.

For it is the join of two planes, or of a plane and the straight line. [Prop. 13.

Def. 2. The **orthogonal projection** of a point **on a straight line** is the foot of the perpendicular drawn through the point and cutting the straight line.

Examples III.

1. If a straight line be rotated about a co planar axis from a position AB into a position A′B′, the two straight lines AB, A′B′ are co-planar.

* Euc. xi. 6.

2. If one of two skew straight lines is turned about the other from a position AB into a position A′B′, the two lines AB, A′B′ are skew, unless AB is perpendicular to the axis.

3. The extremities of one diagonal of a parallelogram are equidistant from any plane which contains the other diagonal.

4. If two plane figures are in perspective, they remain so when either is turned about their common join as axis.

5. ABC is a triangle and O is a point outside its plane. If OD be the normal from O on the plane and OE the perpendicular on AB, show that DE is perpendicular to AB. If D be the orthocentre of ABC, show that the normal from B on the plane OAC meets OD.

6. If a plane figure be rotated about any axis lying in its own plane from one position ABC . . . to another A′B′C′ . . ., either of these figures may be regarded as a parallel projection (not orthogonal) of the other.

7. There is in general passing through any point in a given plane one and only one straight line lying in that plane and perpendicular to a given straight line. What is the exceptional case?

8. If two straight lines drawn from a point to a plane are equal, they make equal angles with the normal to the plane, and meet the plane in points equidistant from the orthogonal projection on it of the given point.

9. Of two unequal straight lines drawn from a point to a plane, the greater makes the greater angle with the normal to the plane through the point, and meets the plane in a point at a greater distance from the orthogonal projection of the given point on the plane.

10. Planes through A, B normal to OA, OB respectively meet in the line CD, where D is in the plane AOB; prove that CD is perpendicular to both BD and BA.

11. On every straight line, not parallel to a certain plane, there is one point and only one point equidistant from two given points.

12. If AC, BD are two skew straight lines equal in length, and E, F are the centres of AB, CD respectively, EF is less than AC.

13. The orthogonal projections of two perpendicular straight lines on a plane parallel to one of them are at right angles.

14. The trace of one plane upon another is perpendicular to the orthogonal projection of the normal to the first plane on the second.

15. If planes be drawn through a given straight line, the locus of the orthogonal projections on them of a given point is a circle.

16. If AC, BD are two skew straight lines, and points P, Q are taken on them respectively so that AP, BQ are equal, then PQ is parallel to a fixed plane.

17. The locus of points lying in a given plane such that the difference of the squares of their distances from two fixed points (outside the plane) is constant is a straight line perpendicular to the straight line joining the two given points.

18. In every plane, not parallel to a certain straight line, there is one point and only one equidistant from the vertices of a given triangle.

19. The orthogonal projection of a right angle is obtuse if the plane of projection meets both or neither of its arms, without these lines having to be produced backwards; if one of these lines must be produced backwards to meet the plane and the other need not, the projection of the right angle is an acute angle.

20. If the orthogonal projection of a right angle is a right angle, the plane of projection is parallel to one of its arms.

21. If the algebraic sum of the distances from two given planes be the same for two points A and B it is constant for all points in the straight line AB.

22. If the algebraic sum of the distances from two fixed planes is the same for each of three points A, B, C it is constant for all points in their plane.

23. Assuming that if the sum of the distances from two fixed straight lines of a point lying in their plane be constant, the locus of this point is a straight line equally inclined to the given lines, deduce that, if the sum of the distances of a point from two fixed planes be constant, its locus is a plane equally inclined to the two fixed planes and parallel to their join.

24. If three straight lines, a, b, c, are parallel to a plane, and three straight lines drawn in any given direction cut b and c, c and a, a and b respectively, the greatest of them is equal to the sum of the other two.

16. The Shortest Distance between Two Skew Straight Lines.

Def. The shortest distance between two skew straight lines is the length of the shortest line joining them.

Prop. 14. **There is one and only one straight line cutting at right angles each of two given skew straight lines, and its length is the shortest distance between them.**

1. Let AB, CD be two skew straight lines and let CE be parallel to AB. Let A′ be the orthogonal projection of A on the plane CED and let A′B′ be parallel to AB.

A′B′ is therefore parallel to CE, [Prop. 7, p. 12.

And therefore lies in the plane CED. [Def. 2, p. 7.

Hence CD cuts A′B′ in some point H′. [Post. 3, p. 4.

A′B′ also lies in the plane ABA′, [Def. 2, p. 7.

And is therefore the projection of AB on plane CED. [Prop. 9, p. 17.

Thus H′ is the projection of some point H in AB.

HH′ therefore cuts AB, CD at right angles.

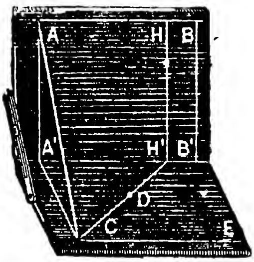
[Def. 1, p. 27; Def. 3, p. 24.

If any other straight line KK′ could cut AB, CD at right angles, it would be parallel to HH′ and the lines AB, CD would be co-planar.

[Prop. 11, p. 24; Prop. 13, p. 27.

Hence HH′ always exists, and is the only straight line cutting AB and CD at right angles.

[Post. 3, p. 4.



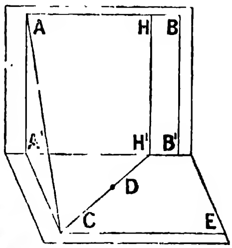


FIG. 14.

2. Let A, C be any two points one on each of the given lines.

The triangle AA′C is thus right-angled at A′;

[Def. 3, p. 24.

Thus AA′ is less than AC. [Euc. i. 16, 19.

But AA′, HH′ being normals, are parallel [Prop. 13, p. 27.

And AB, A′B′ are also parallel, therefore AA′ = HH′.

[Euc. i 34.

Thus HH′ is less than any other line joining the given lines, and its length is the shortest distance between them.

17. The Angle between Two Planes.

So far geometrical magnitudes have been compared by a single superposition. The introduction of the idea of ratio involves comparison by an unlimited number of superpositions if the ratio be incommensurable, and in any case implies arithmetical representation.

We shall take it for granted that all ratios, whether commensurable or incommensurable, obey the same laws.

Def. 1. Two half-planes, each bounded by a straight line which forms their common edge, make two dihedral angles;

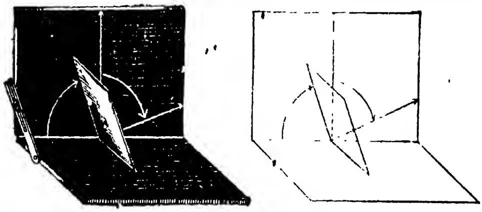


Fig. 15.

their magnitudes being determined by the respective fractions of a complete revolution necessary to move one half-plane from its own position into that of the other by the two alternative routes.

As the half-plane rotates, the normal to it through any point of the axis of rotation rotates in a plane normal to the axis and returns to its original position when the plane does.

Scholium: The magnitude of the dihedral angle may therefore be denoted by the same measure as the plane angle between the two corresponding positions of the normal, or by that of the plane angle between the two positions of a line in the plane, perpendicular to the axis of the dihedral angle.

Def. 2. Two planes are perpendicular when the angles made by them are four right dihedral angles.

PROP. 15. The statements

(i) **two planes are perpendicular,**

(ii) **a straight line of one plane is normal to the other,**

are equivalent.

Let the two planes α, β cut in AB, and let BAC, BAD be right angles in the planes α, β respectively.

Statement (i) of the enunciation makes CAD a right angle. [Def. 2, Sch. p. 32.

Statement (ii) makes a normal to β lie in α.

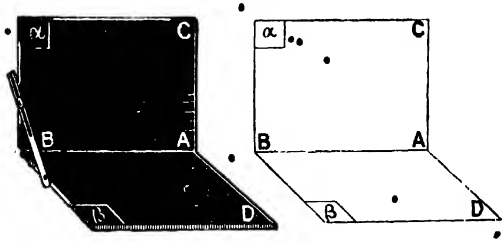


FIG. 16.

But this normal must be perpendicular to AB, and therefore parallel to AC.

Hence, again, the angle CAD is a right angle. [Def. 3, p. 24.

The two statements are therefore equivalent.

† **Scholium:** The join of two planes, each perpendicular to a third plane, is normal to that plane.

For the normal to that plane, through any point common to the first two, lies in each of them and is therefore their join.

* Euc. xi. 18. † Euc. xi. 19.

18. The Angle between a Straight Line and a Plane.

Def. The angle between a straight line and a plane is th angle between the straight line and its orthogonal projectio upon that plane.

Prop. 16. The measure of the acute dihedral angle betwee two planes is that of the greatest acute angle which an straight line in one plane makes with the other plane.

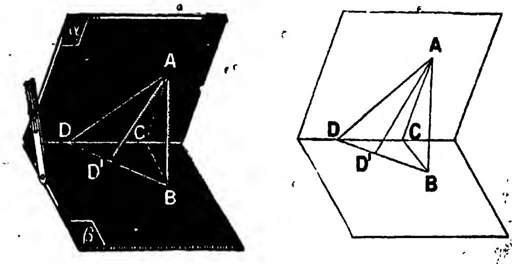


Fig. 17.

Let B be the projection on a plane β of any point A i another plane α. Let the perpendicular from A to the joi $(\alpha\beta)$ meet it in C, and let D be any other point in $(\alpha\beta)$.

Since CD is perpendicular to AC, AB it is normal to th plane ABC, and is therefore perpendicular to CB.

The measure of the acute dihedral angle $(\alpha\beta)$ is the sam as that of the plane angle ACB.

Also, ADB is the angle which AD, any straight line i α, makes with the plane β.

It is required to show that ACB is the greatest val which ADB can assume.

If D is any position other than C, the triangle DCB is right-angled at C, and therefore BD is greater than BC. [Euc. i. 19.

Let BD′ be part of BD equal to BC, the triangles ABC, ABD′ being both right-angled and therefore congruent. [Euc. i. 4.

But the angle AD′B is exterior to the triangle AD′D, and therefore the angles ACB, AD′B are each greater than the angle ADB;

That is, ACB is the greatest angle which any straight line in α makes with β, and the measure of ACB is that of the dihedral angle $(\alpha\beta)$.

Examples IV.

1. The projection of any segment of a straight line on any plane to which it is parallel is of the same length as and parallel to the given segment.

2. The projections of segments of a straight line have to one another the same ratio as these segments.

3. Two dihedral angles which have their edges parallel, and the faces of either respectively perpendicular to the faces of the other, are equal or supplementary.

4. The locus of points equidistant from two planes is the pair of planes bisecting the angles between them.

5. When a straight line is parallel to a plane the shortest distance between it and any straight line to which it is not parallel lying in that plane is constant.

6. The locus of points equidistant from two intersecting straight lines is a pair of perpendicular planes, each of which is equally inclined to the given straight lines.

7. If a plane is equally inclined to two straight lines it is parallel to one of the straight lines bisecting the angles

8. Three parallel planes divide all straight lines which cut them in the same ratio.

9. If a straight line be equally inclined to the sides of a dihedral angle, its traces on those planes are equidistant from their join and conversely.

10. If in a tetrahedron three intersecting edges are mutually perpendicular, the orthogonal projection of any edge upon the opposite edge is a point.

11. Assuming that the locus of points P lying in a plane AOB, such that the sum of the projections of OP on OA and OB is constant, is a straight line, the locus of all such points P in space is a plane perpendicular to the given plane.

12. A, B are the projections of a point P on two perpendicular planes; if the perpendicular distances of A, B from the join of these planes be given, give a plane construction of the perpendicular distance of P from this line.

13. The traces of a straight line on perpendicular planes are A and B; given the lengths of the perpendiculars AA′, BB′ drawn to the join of these planes and the distance A′B′, give plane constructions for the angles which AB makes with the perpendicular planes, and with A′B′.

14. Give a plane construction for the projections on two perpendicular planes of a straight line which makes given angles with them, and which passes through a point the projections of which are known.

15. Give a plane construction to find the inclination of a segment of a straight line to the join of two perpendicular planes, knowing the lengths of its projections on the two planes and the inclination of one projection to their join.

16. Given the orthogonal projections of a point P on two planes at right angles and A, B the traces on these planes of a straight line, find a plane geometrical construction for the perpendicular distance of the given point from the given straight line. (Construct a triangle with sides equal to those of PAB.)

17. The locus of straight lines equally inclined to two intersecting straight lines, and passing through their common point, is a pair of planes perpendicular to the plane containing them.

18. A straight line which is equally inclined to three intersecting straight lines in a plane is normal to that plane.

19. All straight lines equally inclined to two given planes are parallel to one or other of two fixed planes.

20. If two straight lines in one plane be equally inclined to another plane, they are equally inclined to the join of those planes.

21. If AA′, BB′, CC′, DD′ are parallel edges of a parallelepiped, the shortest distance between AC and B′D′ is not greater than that between AC and A′C′.

22. The locus of points dividing in a given ratio any straight line which joins two parallel planes is a parallel plane.

23. The locus of points dividing in a given ratio any straight line which joins two skew straight lines is a plane parallel to both.

24. If two skew straight lines AB, CD are joined by straight lines AC, BD, which are bisected at P, Q respectively, all straight lines which join the two given lines and cut PQ are bisected by it.

25. Through any given point either two straight lines or none can be drawn to meet a given circle, and a given straight line not in the same plane with it. What exceptional cases occur?

26. The locus of points which divide in a given ratio all straight lines parallel to a given plane which join two given skew straight lines is a straight line.

27. If any solid figure is rotated about an axis, from a position denoted by ABC . . . to that denoted by A′B′C′ . . ., the lines AA′, BB′, CC′ are all parallel to a certain plane.

28. A segment of a straight line may be moved from any one position, AB to any other, A′B′ by a rotation about an axis which is perpendicular to both AA′ and BB′.

29. * A solid figure moving with one point fixed at O can be moved from any one position to any other by a rotation about an axis passing through O.

30. † A solid figure can be removed from any one position to any other, either by rotations about two successive axes or by a pure translation ‡ and a rotation.

31. Every line whose orthogonal projections on two intersecting planes are both straight lines is a straight line.

32. If P, Q, R, S are four points, one on each of four parallel edges of a parallelepiped, what is the condition that they should be co-planar if they cut off from the edges lengths a, b, c, d respectively?

33. On either of two skew straight lines there are two points at a given distance from the other line, provided that this distance exceeds the shortest distance between them.

34. Given the projection by parallel lines of a skew quadrilateral ABCD on a plane parallel to AB and CD, find a plane construction for the length and direction of a straight line which is parallel to the same plane, and which divides AD, BC in a given ratio.

35. If in the preceding question A′B′C′D′ is the projection of the skew quadrilateral ABCD and D′A′B′P is a parallelogram, the length and direction of the shortest line which is parallel to the plane of projection and which joins AD and BC, is equal to that of the perpendicular drawn from D to C′P.

36. Joining any two skew straight lines there can be drawn two, and only two, straight lines parallel to a given plane and of given length, provided that this length exceeds a certain value.

* Euler, 1750. † Chasles, 1830.

‡ A pure translation is such that the displacement of any one point of the figure is equal and parallel to that of any other.

SUPPLEMENT.

19. Symmetry.

Def. Two points are **symmetrical with respect to a given centre,** when that centre bisects the straight line which joins them; either point is the symmetric point of the other with respect to the given point.

Def. Two points are **symmetrical with respect to a given axis,** when that axis bisects the straight line which joins them and is perpendicular to it; either point is the symmetric point of the other with respect to the given straight line.

Def. Two points are **symmetrical with respect to a given plane,** when that plane bisects the straight line which joins them and is normal to it; either point is the symmetric point of the other with respect to the given plane.

Def. Two figures are **symmetrical with respect to a given centre, axis, or plane,** when corresponding points are symmetrical; and one figure is the symmetric figure of the other with respect to that centre, axis, or plane.

Def. A figure is **symmetrical with respect to a given centre, axis, or plane,** when it coincides with its symmetric figure.

Prop. 17. Two figures symmetrical with respect to an axis are congruent. (See Fig. 18, p. 10.)

Let AA′, BB′, CC′, be pairs of points symmetrical with respect to an axis l, which bisects AA′, BB′, CC′ in A″, B″, C″ respectively. If, with l as axis of rotation, the points A, B, C be rotated through half a complete revolution, A″A, B″B . . . will coincide with A″A′, B″B′ . . . respectively.

The two figures ABC . . , A′B′C′ . . . are therefore congruent.

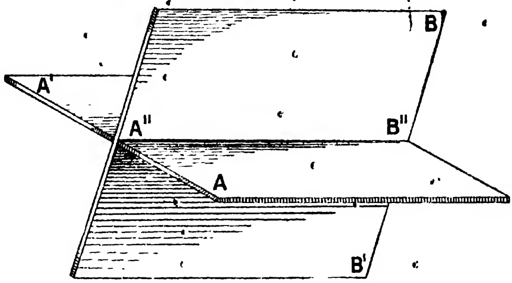


Fig. 18.

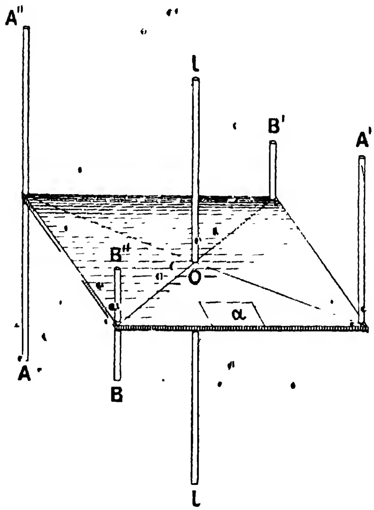


Fig. 19.

PROP. 18. **Two figures symmetrical with respect to a centre can be placed so as to be symmetrical with respect to a plane, and conversely.**

Let l be any straight line through O the centre of symmetry, and let α be the plane through O normal to l.

Suppose the figure A'B'C' . . ., the symmetric figure with respect to O of the figure ABC . . ., to be rotated about l as axis through half a complete revolution into a position A''B''C'' . . . Then OA'' lies in the plane containing l and OA, and OA, OA'' are equally inclined to l. Hence AA'' is parallel to l and is therefore normal to α and bisected by it.

Conversely, if ABC . . . , A''B''C'' . . . are symmetrical with respect to a plane, and any point O be taken in it and the figure A''B''C'' . . . rotated through half a complete revolution about the line l through O normal to α into a position A'B'C' . . ., this figure is the image of ABC . . . with respect to the centre O.

Scholium (*a*): Two figures which are symmetrical with respect to a centre or plane have their corresponding straight lines, plane angles, and dihedral angles equal.

For corresponding straight lines are equal in the case of symmetry with respect to a centre, from Euc. i. 4.

Hence follows the equality of the angles, plane and dihedral, from Euc. i. 8 and superposition.

The extension to symmetry with respect to a plane follows from the preceding proposition.

Scholium (*b*): Symmetric figures of a given figure with respect to parallel planes are congruent.

For one figure may be superposed on the other by a uniform displacement of every point of amount equal to twice the shortest distance between the two planes in the direction normal to them both.

Scholium: Symmetric figures of a given figure with respect to two centres or planes are congruent.

Case i., of two centres, is reduced to Sch. (*b*), p. 41, if the given figure is moved through half a complete revolution about the straight line joining the two centres as axis.

Case ii., of a centre and a plane, is reduced to Sch. (*b*) if one figure is moved through half a complete revolution about the straight line through the centre normal to the plane as axis.

Case iii., of two intersecting planes, is reduced to Case ii. if one figure is moved through half a complete revolution about an axis normal to one of the planes.

Examples V.

1. If a pair of planes or straight lines is symmetrical with respect to an external centre, the planes or straight lines are parallel.

2. Any pair of straight lines is in general symmetrical with respect to one centre only, except in one special case when the number of centres is infinite.

3. A pair of planes is symmetrical with respect to an infinite system of points; what special case arises?

4. Two skew straight lines are symmetrical with respect to an axis, which bisects at right angles the shortest line joining them, and is equally inclined to them.

5. Any pair of straight lines is symmetrical with respect to either of two other straight lines which cut at right angles.

6. If a pair of planes is symmetrical with respect to an axis which cuts either, the planes cut in a straight line which meets the axis at right angles, or coincides with it; or the planes must be parallel to one another and to the axis of symmetry.

7. Two straight lines which are symmetrical with respect to a plane make equal angles with it, and lie in a plane perpendicular to it.

8. Any two co-planar straight lines are, in general, symmetrical with respect to either of two definite planes; but in one special case an infinite number of such planes of symmetry exist.

9. If a plane is equally inclined to two straight lines, it is parallel to one of the straight lines bisecting the angles between them.

10. A parallelepiped can be uniquely divided into two symmetric parts by a plane passing through any given straight line.

11. No figure which has two centres of symmetry can consist of a limited number of points; any such figure is made up of points lying in pairs of parallel straight lines.

12. Any figure which has two parallel planes of symmetry must consist of points lying in straight lines normal to those planes.

13. No figure, which has two planes of symmetry not inclined at some commensurable fraction of a complete revolution, can consist of a limited number of points; any such figure must consist of points lying in circles in parallel planes, and possesses an unlimited number of co-axial planes of symmetry.

14. If a figure has a plane of symmetry, and an axis of symmetry oblique to this plane, it is made up of points lying in pairs of equal parallel circles.

15. If a figure has in all n planes of symmetry which have a common join, to every point there correspond $(2n - 1)$ other points, and the common join is an axis of symmetry when n is an even number.

16. If a figure has a centre and a plane of symmetry, the normal to the plane drawn through the centre is an axis of symmetry.

17. If the angle BAC is bisected by AD, and any straight line PAQ is drawn perpendicular to AD, the angles BAP, CAQ are equal.

18. A, B are two points on the same side of a plane α, and A′, B′ are their symmetrical points with respect to this plane; if P is any point of the plane α, the sum of the distances AP, PB is least for the point in which AB′, A′B meet the plane.

19. A, B are two points on opposite sides of a plane α, and A′, B′ are symmetrical points with respect to this plane; if P is any point of the plane α, the difference of the distances AP, PB is greatest for the point in which AB′, A′B meet the plane.

20. If a system of points has two intersecting axes of symmetry, corresponding sets of points lie on concentric spheres.

21. With any six unequal straight lines as edges, any three of which can be used to form a triangle, two sets of different tetrahedrons, fifteen in each set, can be constructed, and one set contains the symmetric figures of the other set.

CHAPTER III

POLYHEDRA.

20. Trihedral Angles. Face Angles.

Def. Three or more plane angles AOB, BOC, COD . . . meeting successively along OB, OC . . . OA, form a **solid angle** at O; the point O is the **vertex**, the straight lines OA, OB, OC . . . are the **edges**, and the angles AOB, BOC, COA . . . are the **face angles** or **sides**.

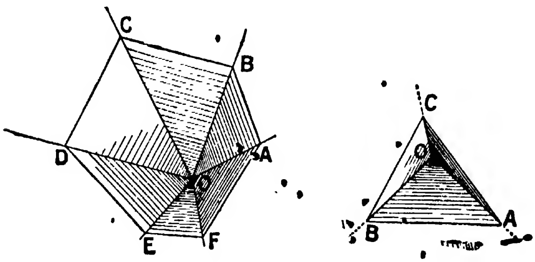


FIG. 20.

Def. A solid angle which has three face angles is a **trihedral angle**; one with four, five . . . is **tetrahedral**, **pentahedral**, . . . or, in all cases, simply **polyhedral**.

Def. A solid angle is **convex** if a plane can be drawn cutting its faces in a convex polygon. All other solid angles are **reflex**.

As a plane can be drawn cutting the faces of a convex trihedral angle in a triangle, each of its face angles must be less than two right angles (Euc. i. 17), and conversely any trihedral angle, the face angles of which are each less than two right angles, is a convex solid angle. A figure of a reflex trihedral angle is given below.

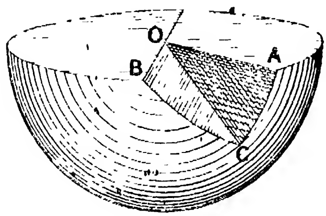


FIG. 21.

Scholium: A trihedral angle is convex if each of its face angles is less than two right angles.

In what follows only convex solid angles will be considered.

*PROP. 19. **In a convex trihedral angle any two face angles are together greater than the third.**

The proposition only requires proof when this third face angle is greater than either of the other two.

Let O, ABC be a trihedral solid angle and let COA be greater than either AOB or BOC.

From AOC let OD cut off an angle AOD equal to AOB and let OD = OB. Let any straight line through D meet OA, OC in A, C respectively.

The triangles OAB, OAD are congruent, [Euc. i. 4.
and therefore AB = AD.

* Euc. xi. 20.

But the sum of AB, BC is greater than the sum of AD, DC; [Euc. i. 20.
and therefore BC is greater than DC.

In the triangles OBC, ODC

OB = OD, OC is common and BC is greater than DC.

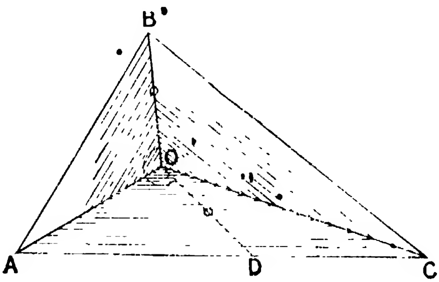


FIG 22.

Therefore BOC is greater than COD. [Euc. i. 25.

Hence the sum of AOB, BOC is greater than the sum of DAO, DOC, that is, than AOC.

21. The Congruence of Solid Angles.

Def. Let O, ABC . . . be any solid angle, and let the edges be produced backwards to form another solid angle O, A′B′C′ . . .; any solid angle which is congruent with one of these is **symmetric** * to the other.

The supplement at the end of the preceding chapter deals with the question of Symmetry (§ 19, p. 39). In particular, symmetry with respect to a plane is shown to imply symmetry with respect to a centre (Prop. 18, p. 41), and such symmetric figures of a given figure are shown to be congruent (Sch. p. 42). The above definition of symmetric solid angles is therefore in accordance with previous statements.

* Legendre (1772-1833). *Eléments de Géométrie.* Paris, 1794.

Scholium: The face angles and dihedral angles of a solid angle are respectively equal to the corresponding angles of any solid angle symmetric to it; but if two corresponding

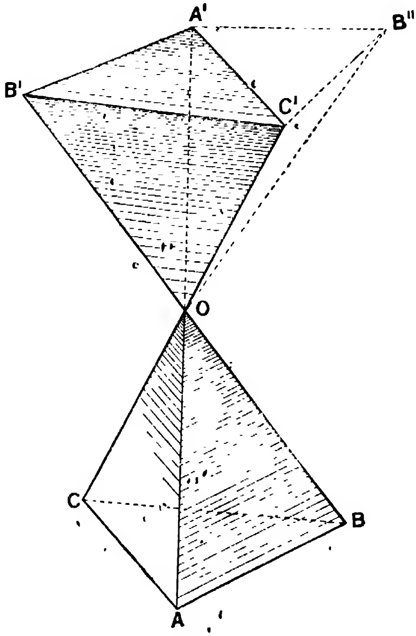


FIG. 23.

edges were superposed, the solid angles would lie on opposite sides of the plane containing them.

PROP. 20. If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, they are congruent or symmetric.

Let OABC, O′A′B′C′ be two solid angles such that AOB = A′O′B′, BOC = B′O′C′, COA = C′O′A′.

Let OA = O′A′, and suppose the planes ABC, A′B′C′ normal to OA, O′A′ respectively.

The triangles OAB, O′A′B′ are congruent; [Euc. i. 26. whence OB = O′B′, AB = A′B′.

Similarly the triangles OAC, O′A′C′ are congruent, and OC = O′C′, AC = A′C′. [Euc. i. 26

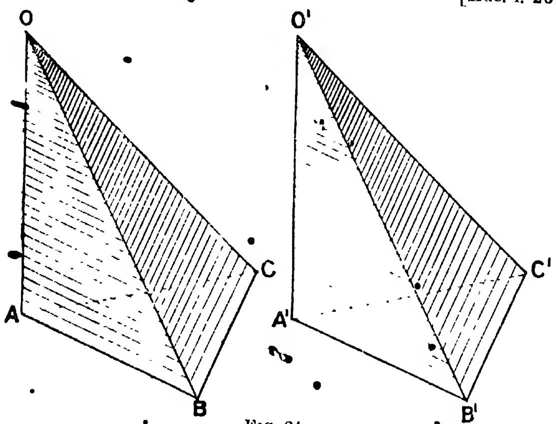


FIG. 24.

Hence the triangles OBC, O′B′C′ are congruent, giving BC = B′C′. [Euc. i. 4.

Hence also the triangles ABC, A′B′C′ are congruent, whence BAC = B′A′C′. [Euc. i. 8.

If O′A′B′C′ is now placed so that O′A′B′ coincides with OAB, B′C′ either coincides with BC (both lying in the plane through A normal to OA) or is symmetric to it with regard to the plane OAB.

The solid angles are therefore congruent or symmetric.

D

Scholium (*a*): Two trihedral angles are congruent or symmetric if they have the following parts of one equal to corresponding parts in the other :

(i) three face angles, [Prop. 20, p. 48.

(ii) two face angles and their included dihedral angle. (Superposition.)

(iii) two dihedral angles and the face angle adjacent to both. (Superposition.)

Two trihedral angles are also congruent or symmetric if they have three dihedral angles of one equal to three dihedral angles of the other ; but the proof is best postponed. The proposition is proved in the supplement to Chapter IV., dealing with spherical geometry (Sch. (*b*), p. 97.)

22. The Prism.

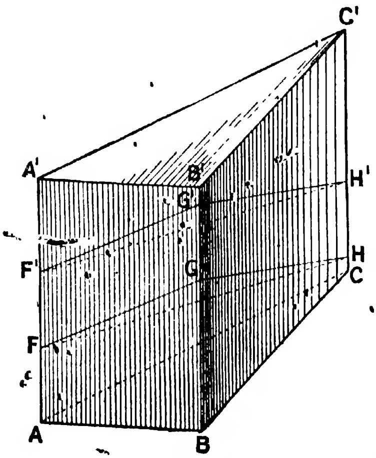


Fig. 25

Def. A **prism** is a solid bounded by two ends which are congruent, parallel, and similarly situated polygons, joined by sides which are the quadrilaterals formed by joining corresponding vertices of the two ends.

Scholium (*b*): The lateral edges of a prism are all equal and parallel, and the lateral faces are parallelograms.

Prop. 21. Sections of a prism by parallel planes are congruent polygons. (See Fig. 25.)

Let AA′, BB′, CC′ . . ., the lateral edges of a prism, cut two parallel planes in points F, G, H . . . and F′, G′, H′ . . . respectively.

The figure FGG′F′ is a parallelogram, [Prop. 3, p. 8.
and therefore FG = F′G′. [Euc. i. 34.

Similarly GH = G′H′; and since FG, GH are respectively parallel to F′G′, G′H′, [Prop. 3, p. 8.
we have FGH = F′G′H′. [Prop. 10, p. 23.

Hence the two polygons FGH . . ., F′G′H′ . . . have their corresponding sides and angles equal, and are therefore congruent.

Def. A **parallelepiped** is a prism of which the ends as well as the sides are parallelograms.

Def. Parallel faces of a parallelepiped are **opposite**; the faces opposite to the two faces containing any edge cut in the opposite edge; the faces opposite to the three faces containing any vertex cut in the opposite vertex; the straight line joining opposite vertices is a **diagonal**.

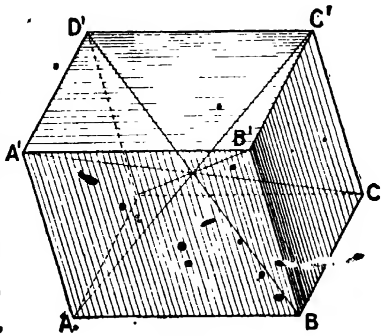


Fig. 26.

Scholium: The four diagonals of a parallelepiped are concurrent and bisect one another.

For any two diagonals of a parallelepiped are diagonals of the parallelogram formed by joining the ends of a pair of opposite edges of the parallelepiped.

Examples VI.

1. Through any point of space there passes one, and only one, plane which cuts the faces of a tetrahedral angle, produced if necessary, in a parallelogram.

2. If two face angles of a trihedral solid angle are equal the opposite dihedral angles are equal and conversely.

3. If any point within a trihedral angle be joined to the vertex, the sum of the angles which this straight line makes with the three edges lies between the sum and half the sum of the face angles.

4. If a trihedral angle has one right dihedral angle, any plane normal to one of its edges cuts it in a right-angled triangle.

5. There are four, and only four, straight lines which pass through the vertex of a trihedral angle and are equally inclined to its three faces.

6. There are four, and only four, straight lines which pass through the vertex of a trihedral angle and are equally inclined to its three edges.

7. The three planes bisecting the interior dihedral angles of a trihedral angle have a straight line in common, as also the bisectors of one interior and the other two exterior angles.

8. In every trihedral angle the planes drawn through the bisector of each face angle perpendicular to that face have a common join.

9. Prove that in a trihedral solid angle the face angle opposite to the greater of two dihedral angles is greater than that opposite the less and conversely.

10. The sum of the angles formed by the edges of a trihedral angle with the opposite faces is less than the sum of its face angles.

11. In every trihedral angle the planes through each edge and the bisector of the opposite face angle have a common join.

12. In every trihedral angle the planes through each edge perpendicular to the opposite face have a common join.

(Consider their traces on a plane normal to one edge of the solid angle.)

13. If in each face of a trihedral angle a straight line is drawn perpendicular to the opposite edge, the three straight lines so drawn are parallel to a plane.

14. Two prisms are congruent if either

(i) one end and two adjacent sides,

(ii) one end, an adjacent side, and the dihedral angle between them,

of one are congruent with the corresponding parts of the other and similarly placed.

15. Classify the various forms assumed by a plane section of a parallelepiped, supposing the plane of the section always to remain parallel to its initial position.

16. In a rectangular parallelepiped the sum of the squares on three intersecting edges is equal to the square on a diagonal.

17. A parallelepiped has a centre of symmetry at the point of intersection of its four diagonals.

18. A rectangular parallelepiped possesses three axes, three planes, and one centre of symmetry.

19. A parallelepiped is completely determined if it is known that three of its edges lie in three given straight lines, no two of which are co-planar.

20. In any parallelepiped the sum of the squares on the four diagonals is equal to the sum of the squares on the twelve edges.

21. The straight line joining the centroids of any two sections of a triangular prism is parallel to its lateral edges.

22. The four diagonals of a quadrangular prism can be separated into two pairs of mutually bisecting straight lines, such that the distance between the mid-points of the two pairs is equal to that between the mid-points of the diagonals of the end face.

23. The sum of the squares on the edges of a quadrangular prism exceeds the sum of the squares on the four diagonals by eight times the square of the straight line joining the mid-points of the two pairs of diagonals.

23. The Tetrahedron.

Def. A **tetrahedron** is a solid figure bounded by four plane faces.

Scholium: A tetrahedron has four triangular faces, six edges, each of which is coplanar with all except one of the remaining edges, and four vertices.

Def. Opposite edges of a tetrahedron are edges which do not meet. The six edges of a tetrahedron fall into three pairs of opposite edges.

Prop. 22. **The four mid-points of two pairs of opposite edges of a tetrahedron form the vertices of a parallelogram with sides parallel to the two remaining edges.**

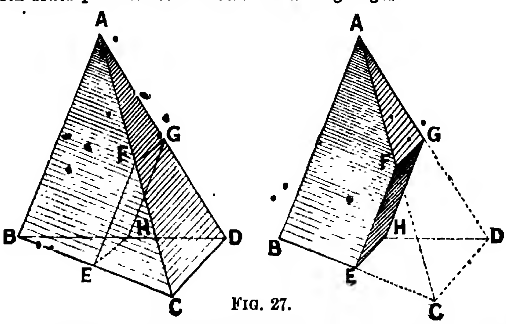


Fig. 27.

Let ABCD be a tetrahedron, and let E, F, G, H be the mid-points of BC, CA, AD, BD respectively.

Because E, F are the mid-points of BC, CA, the lines EF, AB are parallel, [Euc. vi. 2.

and similarly the lines GH, AB are parallel, [Euc. vi. 2.

whence the lines EF, GH are parallel. [Prop. 7, p. 12.

In the same way it follows that

GF, EH, CD are all parallel,

that is, EFGH is a parallelogram with its sides parallel to AB, CD.

Def. The centroid of a triangle is the point common to its three medians; it divides each in the ratio 1 : 2.

Def. The straight line joining any vertex of a tetrahedron to the centroid of the opposite face is a median of the tetrahedron.

Prop. 23. The four median lines meet in a point which divides each in the ratio 1 : 3. (See Fig. 28, p. 56.)

Let ABCD be a tetrahedron, and let E be the mid-point of BC. Let F, H divide ED, EA so that $EF : FD = EH : HA = 1 : 2$, and let DH, AF cut at G.

In order to prove the four medians concurrent in G, it is sufficient to prove that AF is divided at G in the ratio 1 : 3 by any one median HD.

Join HF.

In the similar triangles EFH, EDA—

$$FH : DA = EF : ED = 1 : 3.$$ [Euc. vi. 2, 4.

Hence also in the similar triangles HFG, ADG—

$$FG : GA = FH : DA = 1 : 3.$$ [Euc. vi. 2, 4.

The four medians have therefore a common point G dividing each in the ratio 1 : 3.

Def. The **centroid** of a tetrahedron is the point common to the four medians; it divides each in the ratio 1 : 3.

Scholium (*a*): The centroid of a tetrahedron bisects each of the straight lines joining the mid-points of opposite edges.

For denoting these lines by p, q, r respectively, any one, p say, is co-planar with each of the medians, and therefore cuts all four. The medians are not co-planar and

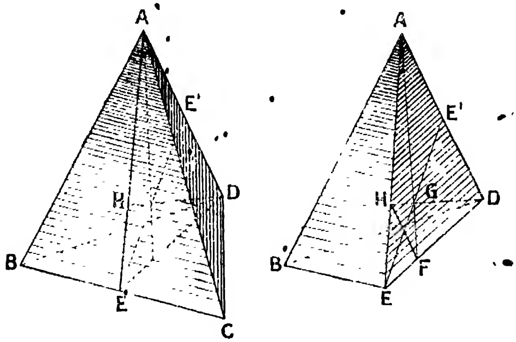


FIG. 33.

have one common point, the centroid G; hence p, q, r all pass through G. But p, q, r bisect one another (Prop. 22, p. 54), and are therefore bisected by G.

Scholium (*b*): Any tetrahedron may be regarded as inscribed within a parallelepiped, so that pairs of opposite edges of the tetrahedron are non-parallel diagonals of opposite faces of the parallelepiped. [See Fig. 29.

The circumscribed parallelepiped is often of great use in proving theorems connected with the tetrahedron.

Scholium (*c*): If two pairs of opposite edges of a tetrahedron are perpendicular, the third pair is also perpendicular.

(Use either (i) The diagonals of a rectangle are equal, and Prop. 22, p. 51.

or (ii) A parallelogram whose diagonals are at right angles is a rhombus, and Sch. (*b*).

Def. If A, B, C, D are any four points **not all in the same plane**, the straight lines AB, BC, CD, DA are said to

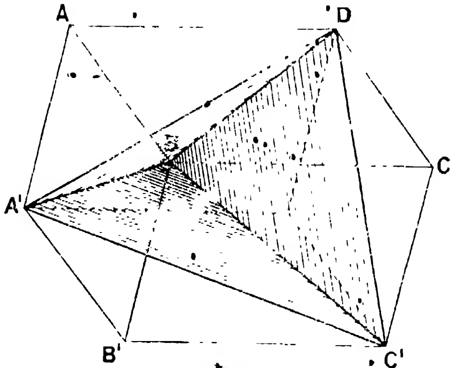


FIG. 29.

be the sides of **a skew** (or **gauche**) **quadrilateral** ABCD; AC and BD are called its **diagonals.**

Scholium: The sides of a skew quadrilateral, together with the two diagonals, form the edges of a tetrahedron.

24. The Pyramid.

Def. A **pyramid** is a solid figure bounded by a **polygonal base** and **triangular sides with a common vertex.**

PROP. 24. **If the sides of a pyramid be cut by a plane parallel to its base, the section is a polygon similar to the base, and is of dimensions proportional to its distance from the vertex.**

Let a pyramid OABCD have a plane section A′B′C′D′ parallel to its base, and let OH, the normal to the base from the vertex, meet the section in H′.

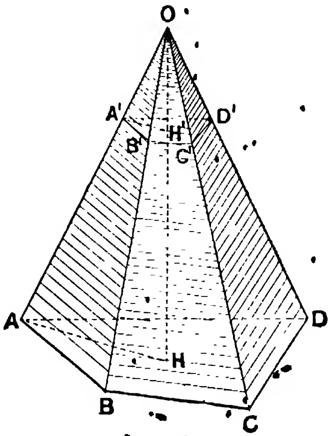


FIG. 30.

Since the lines A′B′, AB are parallel, [Prop. 3, p. 8.

the triangles OA′B′, OAB are equiangular, [Euc. i. 29.

and

A′B′ : AB = OA′ : OA. [Euc. vi. 4.

Similarly,

OA′ : OA = OH′ : OH,

that is,

A′B′ : AB = OH′ : OH.

Similarly,

B′C′ : BC = OH′ : OH,

and A′B′, B′C′ being respectively parallel to AB, BC,

A′B′C′ = ABC, and so also for the other angles. [Prop. 10, p. 23.

The polygons A′B′C′D′ . . ., ABCD . . . are therefore similar, and the dimensions of A′B′C′D′ . . . are proportional to OH′, the distance from the vertex.

25. The Polyhedron.

*** Prop. 25. The sum of the face angles of a convex polyhedral angle is less than four right angles.**

Let any solid angle, vertex O, be cut by a plane in a convex polygon ABCD. . .

Let K be any point within the polygon.

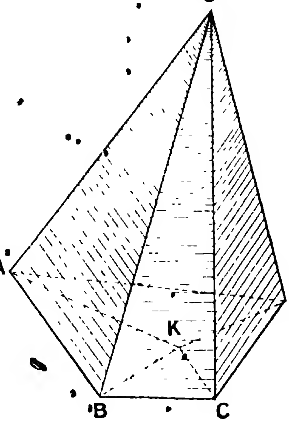


Fig. 31.

The sum of the angles in the triangles OAB, OBC, OCD . . . is equal to the sum of the angles in the triangles KAB, KBC, KCD . . . [Euc. i. 32.

But since the solid angles at A, B, C . . . are all convex trihedral angles, [Sch., p. 46.

and the sum of any two is greater than the third; [Prop. 19, p. 46.

it follows that the sum of the base angles of the triangles OAB, OBC, OCD . . . is greater than the sum of the base angles of the triangles KAB, KBC, KCD . . .

Hence the sum of the remaining face angles at the vertex O is less than the sum of the angles at K, that is, is less than four right angles.

Def. A regular solid angle is one which has all its face angles equal and all its dihedral angles equal.

* Euc. xi. 21 proves this theorem for a trihedral angle.

Def. A **polyhedron** is a solid figure bounded by plane faces; consecutive faces cut in **edges**, and consecutive edges in **vertices** of the polyhedron.

Def. A **convex** polyhedron is one which lies entirely on one side of the plane containing any face.

Def. A **regular polyhedron** is one which has all its faces regular and congruent, and all its solid angles regular and congruent.

***PROP. 26. There cannot be more than five regular convex polyhedra.**

A regular polyhedron is completely determined if m, the number of edges meeting in each solid angle, and n, the number of edges in each of the faces, are both known.

The values of m, n are restricted by the fact that the sum of the face angles of each solid angle must be less than four right angles. [Prop. 25.

Since the least value of m is three, the largest face angle allowable is less than 120°. The faces must therefore be triangles, squares, or pentagons.

In the first case the only possible values of m are 3, 4, 5; in the second and third cases the only possible value of m is 3.

There are, therefore, not more than five regular convex polyhedra.

* Euc. xiii., the concluding Scholium.

Regular polyhedra can be constructed in each of the five ways indicated in the last proposition. Card models may be made by folding the following figures along the dotted lines and placing together points marked with the same letter. The small rectangular flaps are for gumming the edges together.

(i) The regular tetrahedron has three equilateral triangles meeting at each vertex, four faces, and four vertices.

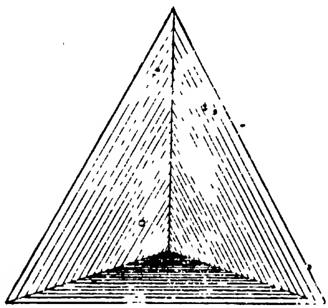


Fig. 32.

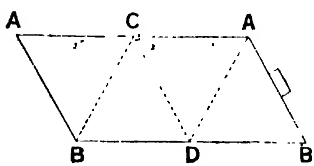


Fig. 33.

(ii) The regular octahedron has four equilateral triangles meeting at each vertex, eight faces, and six vertices.

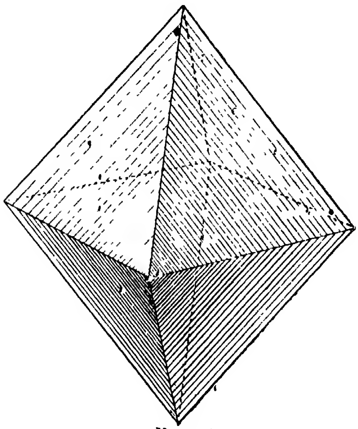


FIG. 34.

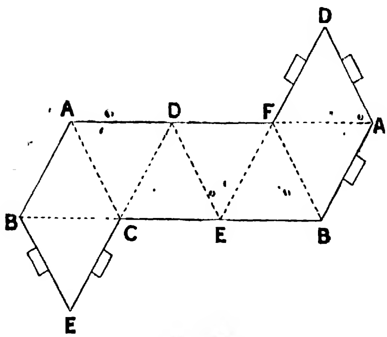


FIG. 35.

(iii) The regular icosahedron has five equilateral triangles meeting at each vertex, twenty faces, and twelve vertices.

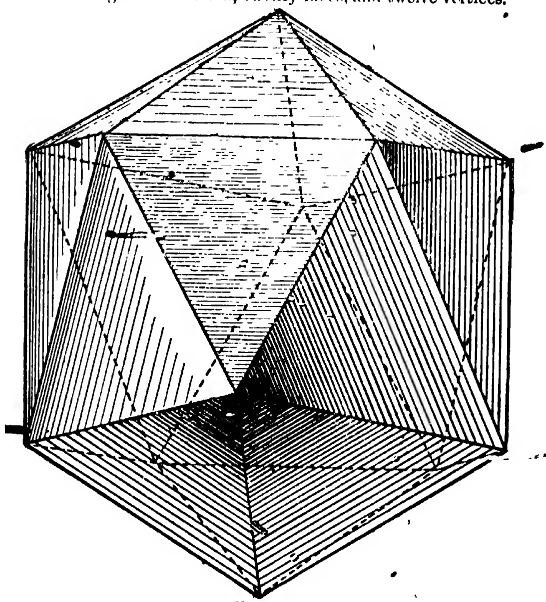


FIG. 36.

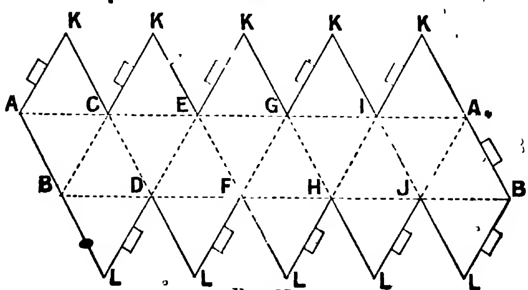


FIG. 37.

(iv) The regular hexahedron or cube has three squares meeting at each vertex, six faces, and eight vertices.

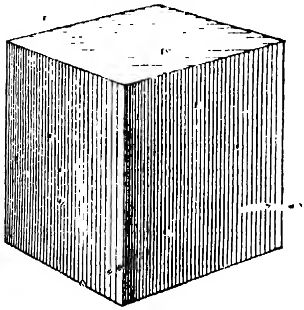


FIG. 38.

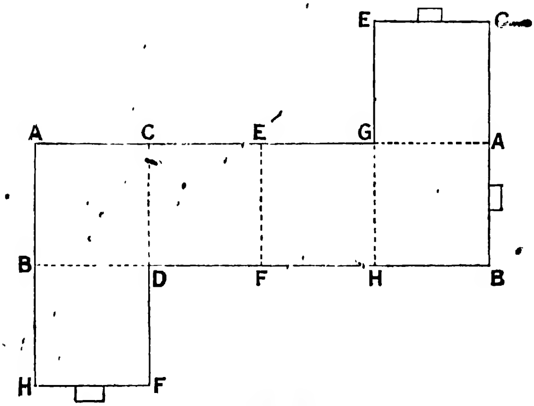


FIG. 39.

(v) The regular dodecahedron has three regular pentagons meeting at each vertex, twelve faces, and twenty vertices.

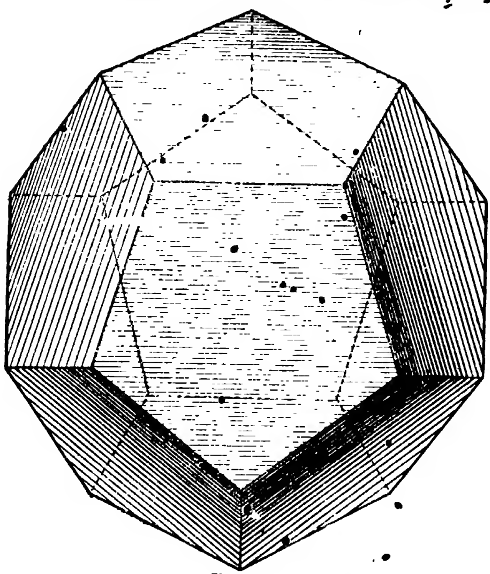


FIG 40.

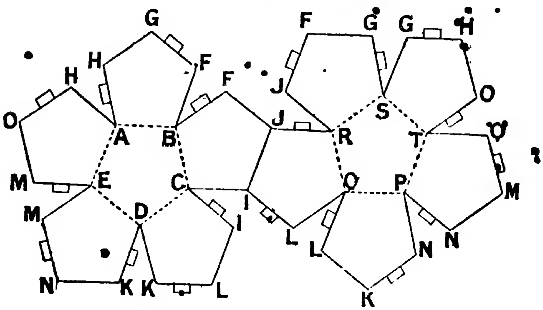


FIG. 41.

The regular solids held an important place in the early development of geometry owing to the mystical significance which was attached to them. They were known to the Pythagorean school (500 B.C.), and were later associated with the name of Plato (429-348 B.C.) by being called "Platonic bodies." The formal demonstration of their existence, by describing their actual construction, is given at the end of the last book (XIII.) of the Elements of Euclid (330-275 B.C.).

Examples VII.

1. There is one parallelepiped circumscribing a given tetrahedron; show how it may be determined. [Fig. 29, p. 57.

2. Every parallelepiped has two inscribed tetrahedrons; show how when one is given the other may be determined.

3. Given the projection by parallel lines of a tetrahedron, determine that of its circumscribed parallelepiped.

4. Opposite edges of a regular tetrahedron are perpendicular.

5. If AP is a normal to the face BCD of a regular tetrahedron ABCD, the length of the normal drawn from P to each of the opposite faces is one-third that of AP.

6. Two tetrahedra are congruent if either

(i) three faces,

(ii) two faces and the dihedral angle between them,

(iii) one face and three dihedral angles,

or (iv) one edge and four dihedral angles

of one are congruent with the corresponding parts of the other, and similarly placed.

7. At each vertex of a tetrahedron in which opposite edges are equal, the sum of the face angles is two right angles.

8. If a tetrahedron be such that each edge is equal to the opposite edge, all the angles of the faces are acute.

9. Show how to fold a triangular piece of paper so as to form a tetrahedron in which the opposite edges are equal. Under what conditions is this possible?

10. A tetrahedron is uniquely determined when the lengths of the straight lines joining the mid-points of opposite edges and the angles between these three lines are known.

11. Show that the shortest distance between two opposite edges of a tetrahedron is perpendicular to either of the straight lines joining the mid points of the other pairs of opposite edges.

12. The six planes which bisect the dihedral angles of a tetrahedron meet in a point.

13. The normals drawn to the faces of a tetrahedron through the centres of the circumscribed circles are concurrent.

14. The six planes drawn normally through the mid-points of the edges of a given tetrahedron meet in one point.

15. The three straight lines which join the mid-points of the two pairs of opposite sides, and of the pair of diagonals of a skew quadrilateral, bisect one another.

16. If each edge of a tetrahedron be perpendicular to the opposite edge, show that:

(i) The line joining any vertex to the orthocentre of the opposite face is normal to that face.

(ii) The lines drawn from the vertices normal to the opposite faces are concurrent.

17. In a tetrahedron in which opposite edges are perpendicular, the sum of the dihedral angles between two faces, and the two angles which they make with that edge which cuts both, is two right angles.

18. In a tetrahedron in which opposite edges are perpendicular, the sum of the six dihedral angles between the faces, and the twelve dihedral angles which the edges make with those faces which they cut, is twelve right angles.

19. If in a tetrahedron ABCD the edges AB, CD be perpendicular, show that the normals from A and B to the faces BCD, ACD intersect on the shortest straight line joining AB and CD.

[Hence, if all the opposite edges of a tetrahedron be perpendicular, the shortest straight lines joining them intersect in the same point as the normals from the vertices to the opposite faces.]

20. In a tetrahedron in which opposite edges are perpendicular, the sum of the squares of each pair of opposite edges is the same.

21. If a plane cuts the sides AB, BC, CD, DA of a skew quadrilateral ABCD in points P, Q, R, S, then is

$$\frac{PA}{PB} \cdot \frac{QB}{QC} \cdot \frac{RC}{RD} \cdot \frac{SD}{SA} = $$

22. If a straight line PR divides the opposite sides AB, CD of a skew quadrilateral, so that

$$\frac{PA}{PB} = \lambda \frac{RD}{RC},$$

PR must intersect or be parallel to any other straight line QS which divides BC, DA, so that

$$\frac{QC}{QB} = \lambda \frac{SD}{SA}.$$

23. If OA, OB, OC are three mutually perpendicular straight lines, and if the perpendicular to BC, drawn through A, meet the circle in the plane ABC on BC as diameter in a point P, then PB = OB, PC = OC.

24. Hence, show how a plane may be determined such that its section, with three mutually perpendicular straight lines OA, OB, OC, may be the vertices of a triangle congruent with a given triangle.

25. If P, Q are the mid-points of AB, CD opposite edges of a tetrahedron, and any plane through PQ meets AC, BD in R, S respectively, R, S divide AC, BD respectively in the same ratio.

26. AB is the shortest straight line between AC, BD, also AC, BD are of equal lengths. Show that the shortest straight line joining AB and CD bisects both.

27. If opposite edges of a tetrahedron are equal, the opposite pairs of edges are bisected by the shortest straight lines joining them.

28. Give a plane construction to find the length of the normal drawn from one vertex of a tetrahedron to the opposite face, given the lengths of the six edges.

29. Two pyramids are congruent if either

(i) the base and two lateral faces,

or (ii) the base and one lateral face, and the dihedral angle between them

of one are congruent with the corresponding parts of the other and similarly placed.

30. The intersections of corresponding edges of the two end faces of a truncated prism or pyramid lie in one straight line.

31. A regular tetrahedron has six planes of symmetry, but no centre or axis of symmetry.

32. A regular tetrahedron may be divided up into four regular tetrahedrons and a regular octohedron.

33. OABC is a regular tetrahedron, and O′ is the middle point of the normal from O to the face ABC. Prove that O′A, O′B, O′C are mutually perpendicular.

34. There are four planes, each of which cuts the surface of a given cube, in a regular hexagon.

35. Two tetrahedrons are similar

(i) when they have a dihedral angle of one equal to one angle of the other, and the faces containing it similar and similarly situated to those of the other,

(ii) when they have a face of one similar to a face of the other, and the dihedral angles adjacent to it equal to the corresponding angles of the other.

36. Two tetrahedrons are similar

(i) if three faces of one are similar, and similarly placed to three faces of the other,

(ii) if five dihedral angles of one are equal to the corresponding angles of the other.

37. If the centroids of the faces of a given tetrahedron be joined they form a new tetrahedron which is similar to the symmetric figure of the given tetrahedron.

38. A tetrahedron is uniquely determined when its base and the distances of the mid-points of pairs of opposite edges are known.

SUPPLEMENT.

26. Euler's Theorem.

Euler's theorem* is a general theorem which states that in any polyhedron the sum of the numbers of faces and vertices is greater by two than the number of edges, provided that the polyhedron does not violate either of two restrictions.

The restrictions may be indicated by saying that

(i) no face must be ring-shaped,

and (ii) the solid, as a whole, must not be ring-shaped.

Two typical examples of the figures to be excluded are given opposite.

The first is formed by placing a smaller box on the top of a larger one, the second is a rectangular frame made of a moulding, of which the cross section is not a rectangle but a trapezium.

We may denote the numbers of faces, vertices, and edges possessed by a polyhedron by the letters F, V, E respectively.

* *Mémoires de Pétersbourg*, 1758.

For a simple rectangular block we should have

$$F+V-E=6+8-12=2, \text{ or } F+V=E+2.$$

For the polyhedron represented in the first of the two figures above we should have

$$F+V-E=11+16-24=3;$$

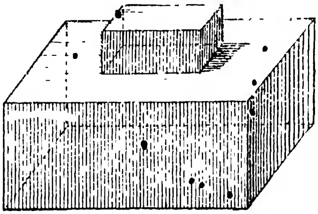


FIG. 42.

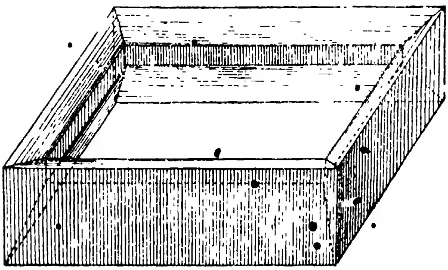


FIG. 43.

and for that in the second of the two figures

$$F+V-E=16+16-32=0.$$

The theorem may be proved by considering a model of the polyhedron, which is constructed by fastening together F polygonal faces, and supposing this model to be taken to pieces, face by face.

At any stage in the process the edges and vertices in any face must be either (i) in contact with some other face or (ii) free; it is only when free edges or vertices are removed that the number of edges or vertices belonging to the figure is affected. Two edges do not meet in a free vertex, unless both are free.

The first face has no free edges or vertices, and when it is removed the numbers of edges and vertices are unchanged.

The second face may be chosen to have one free edge, and there are no free vertices as yet; the number of edges is therefore decreased by one when this face is removed.

The third and following faces may be chosen so that each has at least one free edge. It is now assumed that at any stage of the removal, until only one face is left, (i) all the free edges do not lie in any one face, (ii) the free edges in any face which is to be removed are all connected. We may combine these statements by saying that, in any face which is to be removed, the free edges form an open string.

The vertices at the two ends of this string are not free, and therefore the number of free edges is always one more than the number of free vertices.

Hence, in general, every time a face is removed the number of edges is reduced by one more than the number of vertices, and the excess of edges over vertices is therefore decreased by one each time any face, except the first, is removed.

But when only one face is left the excess of edges over vertices has been reduced to zero.

The total number of faces being denoted by F, it follows that the original excess of edges over vertices, that is, $E-V$, has been reduced by unity $F-2$ times, and is then zero.

That is, $E-V=F-2$ and $F+V-E=2$, or $F+V=E+2$.

In the case of the polyhedron in the first of the two figures on p. 71 an exceptional case occurs, supposing all the faces of the smaller box to have been already removed, when the first face is taken from the remainder. In this case the free edges lie all in one face, and we find that

$$E-V=F-3, \text{ or } F+V-E=3.$$

In the case of the polyhedron in the second of the two figures on p. 71, supposing the faces forming the top of the rectangular frame to be removed first, an exceptional case arises when the fourth of these faces is removed. In this face the two free edges are not connected; its removal therefore takes away two edges but leaves the number of vertices unchanged, and the excess of edges over vertices is decreased by two instead of one. Supposing the vertical faces next taken away, the same thing occurs a second time when the first of the faces at the bottom of the frame is removed.

Hence we have $$E - V = F,$$

or $$F + V - E = 0.$$

The extension of this theorem to other cases than the simple one discussed has been worked out by Listing.

A **simply connected surface** is one in which any curve joining two given points can always be continuously deformed into any other curve which joins them, without crossing the boundaries of that surface.

Euler's theorem is applicable to any solid figure the surface of which is simply connected and which is composed of simply connected faces. (For constructing the surface face by face corresponds to continuously deforming a curve joining two points of one face through the boundaries of all the faces in succession until it returns to its original position.)

Scholium: In any polyhedron in which m edges meet in each vertex and in which there are n edges in each face

$$2E = Fn = Vm.$$

For if we add the numbers of edges meeting in each vertex or lying in each face, each edge is counted twice.

These equations may be written

$$\frac{E}{mn} = \frac{F}{2m} = \frac{V}{2n} = x \text{ (say)}.$$

But, making use of Euler's theorem, we have

$$x = \frac{F + V - E}{2n + 2m - mn} = \frac{2}{2n + 2m - mn},$$

whence the following table is obtained :—

	m	h	E	F	V
Tetrahedron	3	3	6	4	4
Cube	3	4	12	6	8
Dodecahedron	3	5	30	12	20
Octahedron	4	3	12	8	6
Icosahedron	5	3	30	20	12

Two polyhedra are conjugate if the number of vertices in the one is equal to the number of faces in the other.

The tetrahedron is self-conjugate, the cube and the octahedron are conjugate, and the dodecahedron and the icosahedron are conjugate.

Also the polyhedron whose vertices are the centres of the faces of a regular polyhedron is the conjugate of that polyhedron.

Examples VIII.

1. The corners of a cube are cut off by planes passing through the mid-points of each set of three coterminous edges; determine F, V, E for the polyhedron so formed.

2. If the faces of a convex polyhedron are all triangular, the number of faces is even, and four less than twice the number of vertices.

3. If the solid angles of a convex polyhedron are all trihedral, the number of vertices is even, and four less than twice the number of faces.

4. The sum of all the angles of all the faces of a convex polyhedron, whose faces are all convex polygons, is double the sum of the angles of a plane convex polygon having the same number of vertices as the polyhedron.

5. Classify all polyhedrons with five vertices according to the nature and number of their faces.

6. It is known that the sum of the plane angles at any vertex of a convex polyhedron falls short of four right angles. Prove that the sum of the deficiencies, for all the vertices, is eight right angles.

Verify this result for each of the regular polyhedra.

7. Let the numbers of faces and vertices of a polyhedron, each containing n edges, be f_n and v_n respectively; then

$$2E = 3f_3 + 4f_4 + 5f_5 + \ldots$$
$$= 3v_3 + 4v_4 + 5v_5 + \ldots$$

8. If at each of the V vertices of a polyhedron there are m_r faces each containing r edges,

$$m_r V = r f_r;$$

with a similar notation

$$n_r F = r v_r.$$

9. In any polyhedron the number of faces each containing an odd number of edges is even.

10. In any polyhedron the number of solid angles each with an odd number of face angles is even.

11. Show that E is never less than $\frac{3}{2}F$ or $\frac{3}{2}V$, whence in a convex polyhedron E is never greater than $3(F-2)$.

12. Show that $3f_3 + 2f_4 + f_5$ is never less than $(6F - 2E)$, and therefore in a convex polyhedron never less than 12, and similarly for $3v_3 + 2v_4 + v_5$.

13. No polyhedron can be formed which does not contain at least four triangular, quadrangular, or pentagonal faces, and at least four trihedral, tetrahedral, or pentahedral solid angles.

14. Show that in a convex polyhedron V is never less than $\frac{1}{2}F + 2$, and never greater than $2(F-2)$; and that F is never less than $\frac{1}{2}V + 2$, and never greater than $2(V-2)$.

15. Show that f_3 is never less than $2(2F - E)$, and v_3 is never less than $2(2V - E)$; whence the number of faces and vertices together, each containing three edges, can never be less than eight.

16. For any open sheet composed of simple polygonal faces,

$$F + V = E + 1.$$

17. If a convex polyhedron be divided into n smaller polyhedra, and F, V, E denote the numbers of faces, vertices, and edges,

$$E + n + 1 = F + V.$$

What is the relation between the number of faces, edges, and vertices within the given polyhedron?

18. If AA', BB', CC' are three equal mutually perpendicular axes bisecting one another at O, it is possible to construct a regular octohedron, tetrahedron, or cube, the faces of which shall meet the axes in the above-mentioned points, if at all. How many regular tetrahedrons can be so constructed?

CHAPTER IV

CURVED SURFACES

27. Surfaces of Revolution.

Def. A surface of revolution is one which can be generated by rotating a plane figure about some straight line in its plane as axis.

Def. The section of a surface of revolution by a plane containing its axis is a **meridian section.** Any meridian section may be regarded as a generating curve of the surface to which it belongs.

Def. The section of a surface of revolution by a plane normal to its axis is a **right section.**

Scholium: The right section of a surface of revolution is a circle traced out by the point in which the generating curve meets the plane of the section.

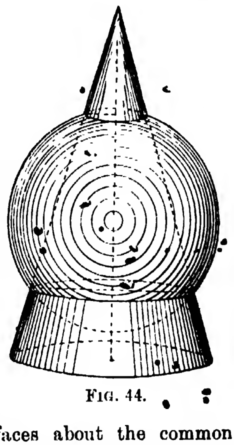


FIG. 44.

PROP. 27. **Two co-axial surfaces of revolution intersect in circles.**

For a curve of intersection is in this case generated by rotating a point of section of co-planar meridian curves of the two surfaces about the common axis; it is therefore a circle in a plane normal to the axis with its centre on that axis.

Def. A **right circular cylinder** is a solid figure formed by rotating a rectangle about one side as axis; the side parallel to the axis is a **generator** of the curved surface, and the other two sides trace out planes forming the **ends** of the cylinder.

Def. A **right circular cone** is a solid figure formed by rotating a right-angled triangle about one of the sides con-

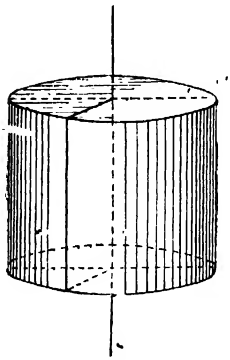


FIG. 45.

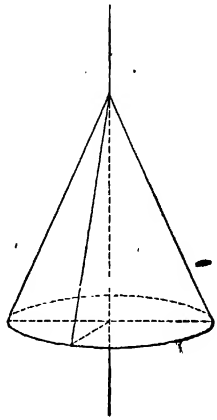


FIG. 46.

taining the right angle as **axis**; the hypotenuse is a **generator** of the curved surface, and the remaining side traces out a plane forming the **base** of the cone. The generators all **meet** the axis in the **vertex** of the cone.

Def. A **sphere** is a solid figure bounded by a surface, all points of which are equidistant from its **centre**; a **diameter** is any straight line passing through the centre and terminated by the surface.

Scholium (*a*): A sphere is a surface of revolution of which any diameter is an axis and of which any meridian section is a circle.

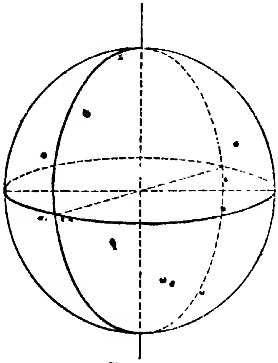


FIG. 47.

Scholium (*b*): The section of any plane with a sphere is a circle. For the diameter normal to the plane is a common axis of revolution for both. [Prop. 27, p. 77.

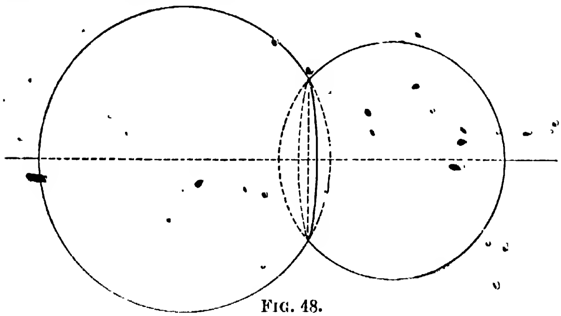


FIG. 48.

Scholium (*c*): The section of two spheres is a circle. For the straight line joining their centres is a common axis of revolution. [Prop. 27, p. 77.

28. Tangent Lines and Planes.

Def. The tangent to a curve.

If A, B be neighbouring points on a curve, the limiting position which is in general approached by the chord AB as the point B approaches A is called the tangent at A.

Def. The tangent line to a surface.

Any straight line through a point on a surface is a tangent line to the surface at that point if it is a tangent to any curve on the surface which passes through that point.

Def. The tangent plane to a surface.

In general the locus of all the tangent lines through a given point of a surface is a plane; this plane is called the tangent plane at the point.

Def. The **normal** to a surface at any point is the normal to the tangent plane at that point.

Prop. 28. The normal at any point of a sphere is the radius through that point.

Take any meridian section of a sphere through a point A on its surface.

This section is a circle with its centre at O, the centre of the sphere, and the tangent at A to this section is a straight line perpendicular to OA.

But all such tangent lines lie in the plane normal to OA; this plane is therefore the tangent plane at A and the radius OA is the normal at A.

Scholium: The tangent lines to a sphere from an external point form a right circular cone. For the straight line AO joining the centre O to the external point A is an axis of

revolution, and any tangent line through A is a tangent to the meridian section containing it; the locus of such

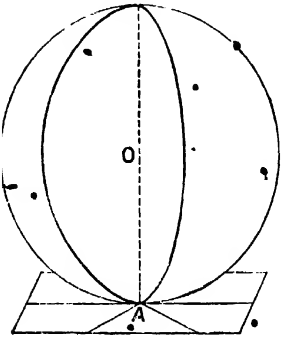


FIG. 49.

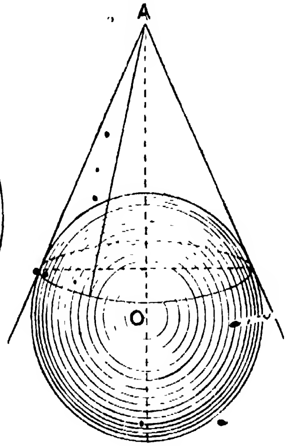


FIG. 50.

tangent lines is therefore the right circular cone formed by the revolution of either tangent, drawn from A to the meridian section, about OA as axis.

Examples IX.

1. There is always one, and only one, sphere which passes through the vertices of a given tetrahedron.

2. The locus of straight lines passing through a given point and making a given angle with a given plane is a right circular cone.

3. If any number of planes be drawn through a given point, the locus of the orthogonal projections upon them of another point is a sphere.

4. If the opposite edges of a tetrahedron be equal, its circumcentre coincides with the point of intersection of the straight lines which join the mid-points of opposite edges.

5. The locus of points in a plane which are such that a given segment of a normal to that plane subtends a given angle at each of them is a circle.

6. If the opposite edges of a tetrahedron be perpendicular, their middle points lie on a sphere which also cuts opposite edges in the feet of the shortest distances between them.

7. The locus of points in a plane such that a given straight line (not in the plane) subtends a right angle there is a circle.

8. The normals drawn through the circum-centres of the faces of a tetrahedron meet in the circum-centre of the tetrahedron.

9. Find the radius of the sphere which can be circumscribed about the frustum of a circular cone in terms of the height of the frustum and the radii of its ends.

$[\frac{1}{2}h^{-1}\{(h^2+r_1^2+r_2^2)^2-4r_1^2r_2^2\}$

10. The centre of a sphere which contains two circles in parallel planes lies between the planes if the square of the distance between the centres of the circles exceeds the difference of the squares of their radii.

11. There are, in general, 2^3 spheres touching the four faces of a given tetrahedron. What exceptions are there?

12. The radii of the circumscribed, escribed, and inscribed spheres of a regular tetrahedron are in the ratios $3:2:1$ respectively.

13. The ratio of the radius of the inscribed sphere to an edge of a regular octahedron is $1:\sqrt{6}$.

14. Two spheres touch at P. Lines APA′, BPB′, CPC′ are drawn terminated by the spheres. Prove that the pyramids ABCP, A′B′C′P are similar.

15. Given any straight line at a distance p from the centre of a sphere of radius a, the tangent plane containing it makes an angle of $\sin^{-1}\frac{a}{p}$ with the plane joining the given line to the centre.

16. The locus of the mid-points of all straight lines of constant length which join two skew perpendicular straight lines is a circle with its centre at the mid-point of the shortest straight line joining them.

17. A fixed point O is joined to any point P in a plane not containing O, and a point Q on OP is taken so that OP . OQ is constant; show that the locus of Q is a sphere.

18. If a continuous figure has two intersecting axes of symmetry inclined at an angle which is an incommensurable fraction of a complete revolution, it is a figure of revolution about an axis through the intersection of the two given axes normal to their plane, which is a plane of symmetry.

19. If any straight line through a fixed point O meet a given sphere in points P, Q, the product of the lengths OP, OQ is constant.

20. Determine the planes passing through a given straight line which cut a given sphere in circles of given radius.

21. The locus of the centres of all sections of a given sphere by planes which pass through a given point is a sphere.

22. The locus of the centres of all sections of a given sphere by planes which pass through a given straight line is a circle.

23. The three edges which meet in one vertex of a tetrahedron are mutually perpendicular; find the radius of the circumscribed sphere in terms of the lengths of these three edges. $[\frac{1}{2}(a^2+b^2+c^2)^{\frac{1}{2}}]$.

24. If two spheres are inscribed in a right circular cone, the plane of their intersection is equidistant from their planes of contact with the cone.

25. Find the radius of a sphere inscribed in a right circular cone of height h and semi-vertical angle θ.

26. The planes passing through the curves of section, two by two, of three spheres have a common join which is normal to the plane containing the centres of the three spheres.

27. The conditions that a sphere can be drawn through two given circles in intersecting planes are that (1) the normals to the planes of the circles through their centres intersect, (2) the tangents drawn to the two circles from any point on the join of their planes are equal.

28. The locus of points in space whose distances from two given points are in a given ratio is a sphere.

29. The locus of points such that the sum of the squares of their distances from two given points is constant is a sphere.

30. The locus of the centres of spheres which cut two given spheres in great circles is a plane normal to the straight line joining centres of the two given spheres.

31. Define the surface which is the locus of points at which a given straight line subtends a constant angle.

Def. The **inverse** of a figure, with respect to a given point O, is the locus of a point P′ lying on the straight line OP, where P is any point of the given figure and P′ is chosen, so that the product OP . OP′ is constant.

The point O is called **the centre of inversion.**

32. The inverse of a sphere, with respect to any point not on its surface, is a sphere.

33. The inverse of a sphere, with respect to any point on its surface, is a plane, which is normal to the diameter through the centre of inversion.

34. Corresponding small parts of a figure and its inverse are approximately similar.

Def. The **stereographic projection** of a figure drawn on a sphere is its inverse, with respect to a point on the surface of that sphere.

35. The stereographic projection of any circle on a sphere is a circle.

36. The angle at which two curves drawn on a sphere cut is equal to that at which their stereographic projections cut.

37. Given a great circle which is its own stereographic projection with respect to a point O, and P′ the projection of a point P, determine Q′, the projection of Q, the antipodal point of P.

[Let Co be the radius of the given circle at right angles to CP′, then Q′ is the point in CP′ produced such that Q′oP′ is right angle.]

38. Given a great circle which is its own stereographic projection and the projections of P_1, P_2, two points on the sphere, determine the circle which is the projection of the great circle through P_1P_2.

39. Given a great circle which is its own stereographic projection and the projection of a point P, determine the circle which is the projection of a small circle of given radius having P as pole.

[With the same notation as in Ex. 37, let oP′ meet the given circle in p, and mark off points r, s, equidistant from p, on its circumference, so that rs is equal to the diameter of the small circle in question. If CP meet or, os in R′, S′, the circle on R′S′ as diameter is the required projection.]

SUPPLEMENT.

29. The Geometry of the Sphere: Definitions and Postulates.

Def. Any plane passing through the centre of a sphere cuts it in a **great circle**; any other plane cuts the sphere in a **small circle**.

The great circle is the fundamental line on a spherical surface, corresponding to the straight line on a plane surface.

Def. 1. The **distance between two points** measured on the sphere is the length of the shorter of the two portions into which they divide the great circle which contains them.

In what follows it will be taken for granted that the "distance" between any two points on a sphere is measured on the surface unless the contrary is stated.

Def. 2. The **angle between two great circles** is the angle between the tangent lines to the great circles at a point common to both.

Scholium: The measure of the angle between two great circles is equal to that of the dihedral angle between their planes. [Sch., p. 32.

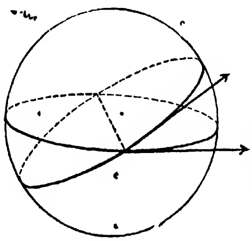


FIG. 51.

For a tangent line to a spherical surface is at right angles to the radius through the point of contact, and if two great circles cut at A their planes cut along the radius OA.

Def. 3. The **unit of length** for distances measured on a sphere is such that the measure of a complete great circle is 2π.

But 2π is the circular measure of the angle subtended by the circumference at the centre. The unit of length is therefore that arc of a great circle which is equal in length to the radius of the sphere.

Def. 4. The **antipodal point** of a given point on a sphere is the opposite extremity of the diameter which passes through that point.

Def. The **antipodal figure** of a given figure is that composed of the antipodal points of all points which that given figure contains.

Def. A **spherical triangle** is the figure formed by the shorter arcs of the great circles which join any three points on the sphere; the given points are the **vertices** of the triangle and the arcs joining them are its **sides**.

Scholium: A spherical triangle is the intersection with the sphere of a trihedral solid angle which has its vertex at the centre; the measures of the **sides** of the spherical triangle are equal to the circular measures of the **face-angles** of the trihedral solid angle, and the measures of the **angles** of the spherical triangle are equal to those of the **dihedral angles** of the trihedral solid angle.

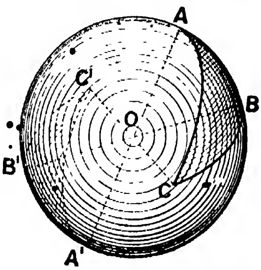


FIG. 52.

As defined above, each side of the spherical triangle must be less than two quadrants,* and therefore the corresponding trihedral solid angle must be convex. [Sch., p. 46.

The postulates on which we base the geometry of the sphere are a selection of four laws which may be shown to contain all others. They themselves may be referred back to the general laws of space, but they contain all that is wanted for the geometry of the sphere, and, if we take them as our starting point, it will be unnecessary to consider any figures which do not all lie in the spherical surface which we are considering. On the one hand the analogy between the geometries of plane and spherical surfaces is thus made more evident, and, on the other hand, when we have obtained the results we require, there is nothing to prevent their translation from expression in

* A quadrant is an arc which is one quarter of a great circle, and therefore subtends a right angle at the centre.

terms of spherical triangles to expression in terms of trihedral solid angles or other convenient forms.

Post. 1. Portions of the same spherical surface may be superposed. [Sch. (*a*), p. 79.

[*Cp.* the principle of superposition in plane geometry.]

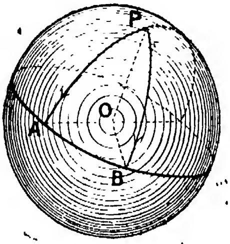


FIG. 53.

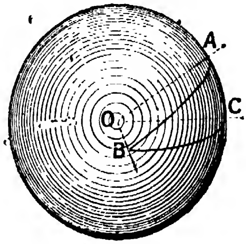


FIG. 54.

Post. 2. Great circles which contain a given point contain its antipodal point also, but have no other points in common. [Def., p. 85.

[*Cp.* Post. 2, p. 4.]

Post. 3. If a point is at the distance of a quadrant from two points of a great circle, one of which is not the antipodal point of the other, it is at the distance of a quadrant from all points upon it. [Prop. 11, p. 24.

Post. 4. Two sides of a spherical triangle are together greater than the third. [Prop. 19, p. 46.

[*Cp.* Euc. i. 20.]

30. The Geometry of the Sphere: Pole and Polar.

Def. Any point and great circle, such that the point is at the distance of a quadrant from each point of the great circle, are called **pole** and **polar** of the other respectively.

Scholium (*a*): Every point has one polar and only one.

For the locus of points at the distance of a quadrant from a given point is the polar of that point. [Post. 3.

Scholium (*b*): If the polar of A passes through B, the polar of B passes through A.

For the condition that either statement should be true is that the arc AB should be a quadrant.

Scholium (*c*): Any great circle has two and only two poles, each the antipodal point of the other.

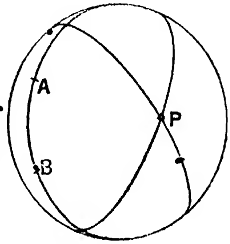


FIG. 55.

For a great circle is determined by two points A, B upon it (Post. 2), and any pole of the great circle AB must lie on each of the polars of A and B (Sch. *b*), which cut in two and only two poles, each the antipodal point of the other. [Post. 2.

The conjugate relation between pole and polar is vitiated by this fact—that each polar has **two** poles, whilst each pole has only **one** polar.

This limitation may be removed if we add the idea of **direction** to that of the great circle. We may then make each pole of a great circle correspond to one of two directions in the great circle.

It is convenient to denote the direction by the order of naming the points which determine the great circle; thus the poles of a great circle AB will be associated with $\overline{AB}$ and $\overline{BA}$ respectively. It is usual to associate with $\overline{AB}$ that pole which lies on our left-hand * side as we walk along the shorter arc AB, outside the sphere, from A towards B.

* This is chosen in conformity with the conventions in a right-handed system of co-ordinate axes.

In the previous figure (p. 89) P is the pole of $\overline{AB}$, P′ that of $\overline{BA}$.

The conjugate relation between pole and polar becomes perfectly definite if we make this distinction.

Def. The **positive direction** round a spherical triangle is such that the area always lies on the left-hand side of a person who walks round the boundary in that direction.

As soon as the idea of direction of a segment of a line is introduced, our arithmetical ideas of addition and subtraction must be generalised into algebraic ones.

Scholium: Considering only points which lie on the same hemisphere, it is easy to verify that in all cases

$$\overline{AB} = \overline{AC} + \overline{CB},$$

where A, B, C are points on one half of a great circle, and $\overline{AB}$, $\overline{AC}$, $\overline{CB}$ denote the distances between the points named, reckoned positive or negative according as the directions indicated, are the same as, or opposite to, the standard direction along that line. The restriction to one half of a great circle is in accordance with Def. I, p. 86.

***Prop. 29. The distance between two points is equal to the distance between their antipodal points.**

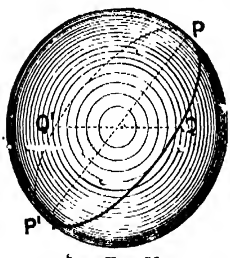


Fig. 56.

Let P′, Q′ be the antipodal points of P and Q respectively.

From Postulates 1, 2 (p. 88), it follows that the complete great circle PQ is bisected by the pair of points P, P′, and also by Q, Q′.

Hence

$$\overline{PQ} + \overline{QP'} = \overline{QP'} + \overline{P'Q'};$$

whence, taking away $\overline{QP'}$ from each side,

$$PQ = P'Q'.$$

* A direct proof follows from the fact that the diameters PP′, QQ′ cut in the centre O.

*Prop. 30. **The measure of the angle between two great circles is equal to that of the corresponding intercept on the polar of their points of intersection.**

Let the great circles meet at P, and let the arcs produced from P meet the polar of P in points A and B.

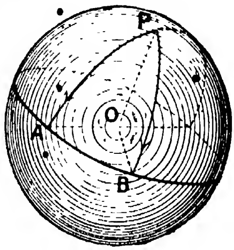


Fig 57.

Since the arc AB may be subdivided into any number of equal parts all subtending equal angles at P, the measure of the arc is proportional to that of the angle. But the measure of a complete revolution is the same for both, and therefore the measures are equal in all cases.

Def. A lune is the portion of the spherical surface intercepted between two half great circles.

†Scholium: Vertically opposite angles are equal, the two angles of a lune are equal, and an angle is equal to its antipodal angle.

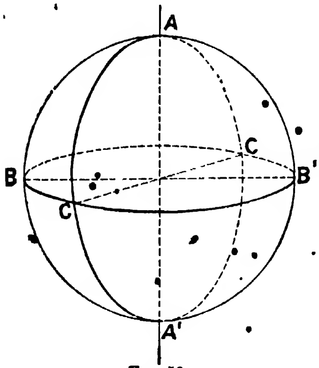


Fig. 58.

Let the antipodal figure of a lune ABA′C be the lune A′B′AC′.

If the polar of A pass through the points B, C, B′, C′, from the last proposition we know that the angles in question have their measures equal to that of BC or B′C′, and are therefore equal. [Prop. 29, p. 90.

* A direct proof follows from the fact that the tangents to great circles at P are respectively parallel to the radii OA, OB.

† A direct proof follows from the fact that the tangents at A, A′ to the two great circles are respectively parallel.

*PROP. 31. **The measure of the angle between two great circles is equal to that of the distance between their poles.**

Let great circles $\overline{AB}$, $\overline{AC}$ have poles P, Q and meet the great circle PQ in points P′, Q′ respectively, such that the quadrantal triangles AP′P, AQ′Q are each named in the positive direction.

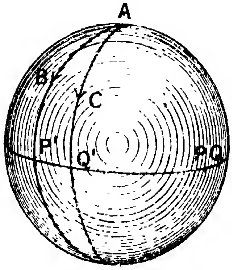


FIG. 59.

Hence, $Q'\overline{Q} = P'P$, and we may choose the letters so that $\overline{PQ}$, $Q'\overline{Q}$ have the same sense, ensuring that Q′PQ are in one semi-circle. But since PQ is less than two quadrants, PQ′ is less than one quadrant, and Q′ lies in the semi-circle of which P is the mid-point.* Hence, P′Q′P lie in one semi-circle, and therefore [Sch., p. 90.

$$Q'P + PQ = Q'Q = P'P = \overline{P'Q'} + Q'P.$$

Taking away $\overline{Q'P}$, $\qquad \overline{PQ} = P'Q'$,

that is $\qquad \overline{PQ} = \overline{BAC}$. [Prop. 30.

31. The Geometry of the Sphere; the Polar Triangle.

Def. The polar triangle of a given triangle has the poles of the sides of the first triangle for its vertices, **each side being taken in the positive direction round the triangle.**† [Def., p. 90.

Scholium: If A′ be the pole of $\overline{BC}$, this being the positive direction round a triangle ABC, A′ is on the same side of BC that A is; that is,

AA′ is less than a quadrant.

* A direct proof follows from the fact that the dihedral angle between two planes has the same measure as the angle between their normals. [Sch., p. 32.

† The choice has been so made that the triangle and its polar are contained in one hemisphere.

This restriction is necessary in order to make the polar triangle a definite thing, for eight $(2 \times 2 \times 2)$ triangles can be formed with poles of the sides of the given triangle as vertices.

PROP. 32. **If A′B′C′ is the polar triangle of BAC, ABC is the polar triangle of A′B′C′.**

For the pole of $\overline{B'C'}$ lies on the polars of B′ and C′ (Sch. (*b*), p. 89), and therefore is at one of the points of section of CA, AB. But the poles A′, B′, C′ have all been so chosen that AA′, BB′, CC′ are each less than a quadrant. That is, A and A′ are on the same side of B′C′; and therefore, in going round A′B′C′ in the positive direction, the poles of the sides are the points A, B, C. This is, therefore, the polar triangle of A′B′C′, if A′B′C′ is the polar triangle of ABC.

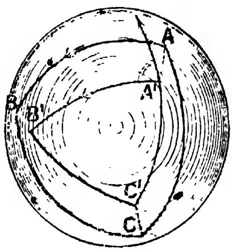


FIG. 60.

PROP. 33. **If a, b, c, A, B, C be used to denote the measures of the sides and angles of a spherical triangle ABC, and a′, b′, c′, A′, B′, C′ those of the corresponding sides and angles of the polar triangle A′ B′ C′, then**

$$a + A' = a' + A - \pi,$$
$$b + B' = b' + B = \pi,$$
$$c + C' = c' + C = \pi.$$

In Proposition 31 it is proved that the measure of the angle between the positive directions $\overline{C'A'}$, $\overline{A'B'}$ is equal to that of the distance BC. But this angle is the exterior angle of the triangle A′B′C′, and is therefore supplementary to the angle A′.

That is, $a + A' = \pi.$

Since ABC is also the polar triangle of A′B′C′, the other equations follow in precisely the same way.

We conclude that any property relating to the sides and angles of a triangle remains true when for each side and angle we substitute the supplement of the corresponding angle or side.

32. The Geometry of the Sphere; the Congruence of Triangles.

The congruence of spherical triangles has in reality, though not in name, been discussed in the section dealing with the congruence of trihedral solid angles (p. 47). The definition there given of the solid angle symmetric with respect to a given solid angle is equivalent to the definition which follows below. For if two spherical triangles ABC, A'B'C' are antipodal, the trihedral angles OABC, OA'B'C' have their corresponding edges continuous.

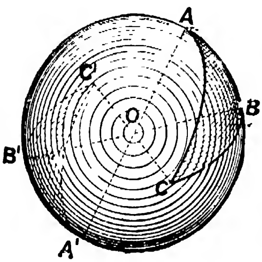


FIG. 61.

Def. Two triangles are **symmetric** if either is congruent with the antipodal triangle of the other.

Since two symmetric solid angles are not congruent, the corresponding spherical triangles are not congruent either—in spite of the fact that the antipodal pairs of sides and angles are separately congruent (Prop. 29, p. 90; Sch., p. 91). The reason is to be found in the fact that if we go round the two triangles, taking the letters in their alphabetical order, the two directions will be found to be opposite to one another, exactly as was found previously in the case of two solid angles.

Def. Parts of two triangles are only said to **completely correspond** when they can be taken in order round the triangles in the **same** direction.

Scholium: Two triangles which have all the parts of one equal to the parts of the other taken round in the opposite direction are symmetric.

Spherical geometry differs from plane geometry in this respect, because, in the latter, two triangles with their parts respectively equal, but not taken round the triangles in the same direction, can be superposed by turning over one of the two triangles and placing it upon the other. But if a spherical triangle be turned over, it can no longer be made to lie on the same spherical surface as before.

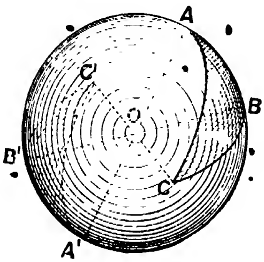


FIG. 62.

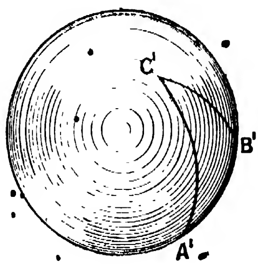


FIG. 63.

The following propositions are divided into two parts, such that each is obtained from the other by replacing the parts of the triangles concerned by the corresponding parts of the polar triangles. The enunciations are generally such that each is obtained from the other by substituting "sides" for "angles," and "angles" for "sides."

PROP. 34. Two triangles are congruent if they have completely corresponding parts equal as follows:—

(i) **Two sides and adjacent angle.**	(ii) **Two angles and adjacent side.**

The proof follows in each case by superposition; in the first case as in Euc. i. 4, in the second as in Euc. i. 26.

*Prop. 35. If in any triangle

(i) two sides are equal, then the angles opposite to them are equal;

(ii) two angles are equal, then the sides opposite to them are equal.

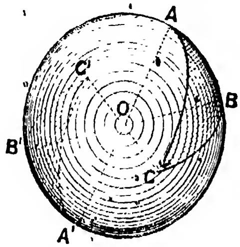


Fig. 64.

Let ABC be any triangle, and A′B′C′ its antipodal triangle.

(i) If AB = AC,
we have

A′B′ = AB = AC = A′C′
[Prop. 29, p. 90.

and BAC = B′A′C′.
[Sch., p. 91.

Hence the triangle ABC has two sides and an angle adjacent to both equal to completely corresponding parts of A′C′B′, and the two are congruent. [Prop. 34.
Whence

∴ ABC = A′C′B′ = ACB.
[Sch., p. 91.

(ii) If ABC = ACB,
we have

A′B′C′ = ABC = ACB = A′C′B′
[Sch., p. 91.

and BC = B′C′.
[Prop. 29, p. 90.

Hence the triangle ABC has two angles and a side adjacent to both, equal to completely corresponding parts of A′C′B′, and the two triangles are congruent.
[Prop. 34.

Whence AB = A′C′ = AC.
[Prop. 29, p. 90.

* *Cp.* Euc. i. 5, 6.

Scholium (*a*): If O is a point on the sphere equidistant from the points A, B, C, the triangles OBC, OCA, OAB may be transposed so as to give the antipodal triangle of ABC by addition or subtraction.

For if O′, A′, B′, C′ be the points antipodal to O, A, B, C respectively, the triangle OBC, being isosceles, is congruent with O′C′B′, and similarly for the other triangles.

*Prop. 36. **Two triangles are congruent, if they have the following completely corresponding parts equal:—**

(i) three sides. **(ii) three angles.**

The proof of the first case follows by the Method of Exhaustion exactly as in Euc. i. 7, 8, or by use of the symmetric triangle of one, as in the more usual proof of Euc. i. 8; that of the second can also be obtained independently, though it is simpler to let the proof depend on that of the first case.

For if two triangles have the angles of one equal to the corresponding angles of the other, their polar triangles are congruent by the first case, whence the given triangles are also congruent.

Scholium (*b*): It may be observed that the second part of this proposition is equivalent to the statement that two trihedral solid angles are congruent if they have their corresponding dihedral angles equal.

Def. A **supplemental** triangle has a pair of sides supplementary and the pair of angles opposite to them supplementary.

* *Cp.* Prop. 20, p. 48. The second proposition is untrue in plane geometry.

PROP 37. **A triangle is supplemental if it has either**

(i) a pair of sides supplementary; **or (ii) a pair of angles supplementary.**

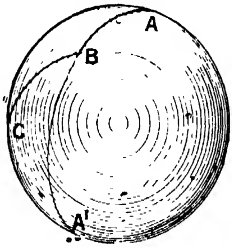


FIG. 65

Let ABC denote the triangle.

(i) Let AB, AC be supplementary and let A, A′ be antipodal points. The triangles ABC, A′CB are congruent by the preceding proposition, since AB + BA′ = AC + CA′ = AB + AC = π.

Whence

AB = CA′, AC = BA′.

Hence the angles at B and C are supplementary.

(ii) Let the angles at B, C be supplementary and let A, A′ be antipodal points. The triangles ABC, A′CB are congruent by the preceding proposition, since the angles at A and A′ are equal,

[Sch., p. 91.

and ACB + A′CB = ABC + A′BC = ACB + ABC = π.

Whence

ABC = A′CB, ACB = A′BC.

Hence the sides AB, AC are supplementary.

* PROP. 38.

(i) The side of a triangle opposite to the greater of two angles is greater than that opposite to the less.

(ii) The angle of a triangle opposite to the greater of two sides is greater than that opposite to the less.

* *Cp.* Euc. i. 19, 18, where the order is reversed, and Ex. 9, p. 52.

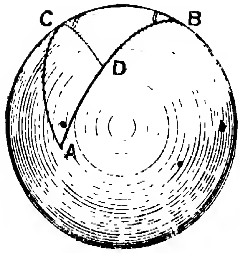


FIG. 66.

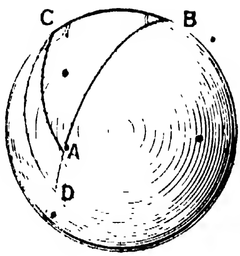


FIG. 67.

(i) For in the triangle ABC let the angle at C be greater than the angle at B. If the angle BCD be equal to ABC, CD falls within the triangle and cuts AD in some internal point D. The angles at B and C in the triangle BCD are equal and therefore BD = CD. [Prop. 35, p. 96.

But CD, DA are together greater than CA.
[Post. 4, p. 88.

Hence BA, the sum of BD, DA, is greater than CA.

(ii) For in the triangle ABC let the side BA be greater than CA. Let the angle BCD be equal to the angle ABC, so that the angle ACB is greater or less than ABC, according as CD falls inside or outside the triangle.

But CD cannot fall outside the triangle, for then AB is the difference between BD, DA, that is between CD, DA (since the triangle BCD is isosceles by Prop. 35, p. 96), which is less than CA (Post. 4, p. 88), contrary to hypothesis. That is, the angle ACB is greater than ABC.

PROP. 39.

* (i) The sum of the sides of a spherical triangle is less than a great circle.

(ii) The sum of the angles of a spherical triangle is greater than two right angles.

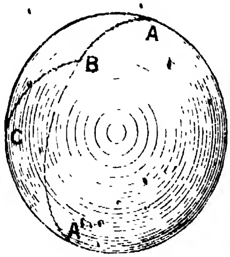


FIG. 68.

(i) Let ABC be any triangle and let A, A′ be antipodal points.

Then BC is less than the sum of BA′, A′C.
[Post. 4, p. 88.

Hence the perimeter is less than the sum of AB, BA′, A′C, CA; that is, is less than a great circle.

(ii) The sum of the sides of the polar triangle of ABC, is less than a great circle.

That is,

$$\pi - A + \pi - B + \pi - C < 2\pi,$$

whence, the sum of the angles A, B, C is greater than two right angles.

33. The Tangent Plane to a Surface of Revolution.

PROP. 40. **The normal at any point, to a surface of revolution is the normal to the meridian section through that point.**

* The first statement is equivalent to Euc. xi. 21 or Prop. 25, p. 59. They were given in the present form along with the preceding pro positions by Menelaus of Alexandria in his work on Spherical Geometry (*circa* 98 A.D.).

Let the normal to the meridian section through A meet the axis in O. Let B be a point which approaches A along some curve drawn on the surface, and let A′ be the point corresponding to A on the meridian section containing B, so that OA′ is a normal to the meridian section through A′B.

The triangle OAA′ is isosceles, and therefore as the angle AOA′ decreases the angles at A, A′ each approach a right angle; that is, the limit approached by the angle between OA′, A′A is a right angle.

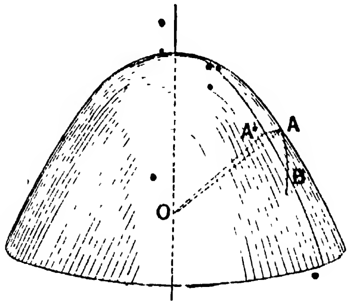


FIG. 69.

Again, since OA′ is the normal at A′, the limit approached by the angle between OA′, A′B is a right angle.

But the limiting position approached by OA′ is the line OA, and therefore the limiting position approached by the plane AA′B is the plane through A normal to OA.

That is, the limiting position approached by the chord AB, which is the tangent at A to an arbitrary curve through A, lies in the plane drawn through A normal to OA; OA is therefore the normal at A to the surface.

Examples, X.

1. Any side of a spherical polygon is less than the sum of the others.

2. If P be any point within a spherical triangle ABC, BA, AC are together greater than BP, PC.

3. If P be any point within a spherical triangle ABC, the sum of the distances AP, BP, CP is less than the whole and greater than half the perimeter of the triangle.

4. The sum of the sides of any convex spherical polygon is less than a complete great circle.

5. The sum of the angles of a convex spherical polygon of n sides lies between $2n$ and $(2n-4)$ right angles.

*6. The bisectors of the angles of a spherical triangle are concurrent.

7. The three perpendiculars to the sides of a triangle drawn through their respective mid-points are concurrent.

8. If one angle of a triangle be equal to the sum of the other two, the centre of the circumscribing (small) circle bisects the greatest side.

9. The bisector of the vertical angle of an isosceles spherical triangle bisects the base at right angles.

10. If a spherical triangle be supplemental, one median bisects the vertical angle and is a quadrant.

11. If the bisector of an angle of a spherical triangle be a quadrant the triangle is supplemental.

12. If one median of a spherical triangle be a quadrant the triangle is supplemental.

13. If DE be an arc of a great circle bisecting the sides AB, AC of a spherical triangle at D and E, P a pole of DE, the angle BPC is equal to twice the angle DPE.

* Menelaus, c. 98 A.D.

14. If in a quadrilateral one diagonal is equally inclined to both pairs of opposite sides, the opposite sides are equal.

15. If the opposite sides of a quadrilateral are equal, its diagonals bisect one another.

16. If in two spherical triangles ABC, A′B′C′, AB, BC are equal to A′B′, B′C′ respectively, and the angle ABC is greater than the angle A′B′C′, the third side AC is greater than A′C′.

17. If in two spherical triangles ABC, A′B′C′, AB, BC are equal to A′B′, B′C′ respectively, and the angles at A, A′ are equal, the angles at C, C′ are equal or supplementary.

18. If in two spherical triangles ABC, A′B′C′ the angles at B and C are respectively equal to the angles at B′ and C′, and the sides AC, A′C′ are also equal, the sides AB, A′B′ are either equal or supplementary. (Contrast Euc. i. 26.)

19. The diagonals of an equilateral quadrilateral bisect one another at right angles.

20. If A′B′C′ be the polar triangle of a triangle ABC and D be the midpoint of the side BC, the polar triangles of the triangles ABD, ACD can be formed by drawing the external bisector of the angle at A′ to meet the base B′C′ in points P and Q.

21. The theorems concerning a spherical triangle and its polar triangle may be extended to any convex polygon.

22. In any spherical triangle, the sides of which are less than quadrants, the exterior angle is greater than either of the interior and opposite angles. (Compare Euc. i. 16.)

23. The sum of two angles of a spherical triangle is greater or less than two right angles according as the sum of the opposite sides is greater or less than two quadrants.

*24. If a quadrilateral be inscribed in a (small) circle the sum of one pair of opposite angles is equal to the sum of the other pair.

* Lexell, 1782.

25. If the opposite sides of a quadrilateral are equal, their intersections lie on the polar of the point of section of the two diagonals.

26. If the base of a triangle be given, and the sum of the base angles exceeds the third by a given amount, the locus of the vertex is a (small) circle passing through the extremities of the base.

27. The great circle which is a tangent to a small circle at any point is at right angles to the radius at that point.

28. The two tangents drawn from any point on a sphere to a small circle are equal.

29. If a quadrilateral circumscribe a small circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

30. If A, A′ are antipodal points, and the small circle inscribed in a triangle A′BC touches A′B at P, AP is one-half the perimeter of the triangle ABC.

31. If s denote the semi-perimeter of a given triangle, and a, b, c the lengths of its sides, the lengths of the tangents drawn to the inscribed circles from the vertices are (s − a), (s − b), (s − c) respectively.

32. The three vertices of a spherical triangle are at the same distance (measured on the sphere) from any great circle which joins the mid-points of two of its sides.

33. The pole of the great circle joining the mid-points of two sides of a spherical triangle is equidistant from the extremities of the third side.

34. The great circle which joins the mid-points of two sides of a spherical triangle cuts the third side in points which are at a distance of a quadrant from the mid-point of the third side.

PART II

MENSURATION

CHAPTER V

MEASUREMENT IN GENERAL

34. The Measurement of Length.

THE process of measuring a straight line consists of two stages:

(i) subdivision into equal parts,
(ii) counting these equal parts.

The subdivision may be either:

(i) of the straight line to be measured,
(ii) of the standard unit in terms of which it is to be expressed,
or (iii) of both.

The equal parts into which the lines are subdivided will be called **elementary** units, to distinguish them from the standard units, such as the foot or centimetre.

As concrete examples of the above consider measurements of 3 in., $\frac{1}{2}$ in., and $\frac{2}{3}$ in. respectively. In the first case the given line is subdivided into three equal parts, each elementary unit being 1 in.; in the second case the standard unit is divided into two equal parts, each elementary unit being $\frac{1}{2}$ in.; and in the third case the standard unit is divided into three and the given line into two equal parts, each elementary unit being $\frac{1}{3}$ in.

But the length of a straight line can only be expressed with perfect numerical accuracy when both it and the standard unit are multiples of some common elementary unit, and in general this is not the case.

In general the process of measuring consists in choosing as elementary unit some sub-multiple of the standard length, say ·01 in. and taking that multiple which lies nearest to the given line as the measure of its length, say 376 hundredths = 3·76 in. The smaller the elementary unit, the more accurate is the measurement.

For practical applications, therefore, it is only necessary to consider such lengths as can be expressed by vulgar fractions (including decimals): that is, all measured lengths are commensurable.

In order to make perfectly general all theorems which concern measurements, incommensurable lengths must also be considered. But, though new definitions of addition, multiplication, and division are required for such numbers, the definitions can be so framed in terms of the corresponding definitions for commensurable numbers that exactly the same laws hold for both.

In all cases, therefore, it is sufficient to deal with lengths as though they were commensurable; the extension to incommensurable lengths having been made once for all.

It should be noted that any indeterminateness about the length of a given straight line is only due to the difficulty of determining and expressing it arithmetically; its actual length is, of course, perfectly definite even when it is incapable of exact numerical representation (viz. when incommensurable).

If the line to be measured is curved instead of straight, the common notion of its "length" is based on the idea of a hypothetical inelastic string which can be superposed upon the curve and then straightened out into a straight line. The numerical measure of the straight line so found is regarded as the length of the curve.

But this notion adds a conception which it is not easy to weld together with our other geometrical ideas, and the necessity for coherence compels us to seek a definition of the length of a curved line in some other form.

The form in which we must frame this definition is that of a limit. Further, the definition presupposes a theorem,* the proof of which belongs to the Integral Calculus. This theorem must now be introduced as a postulate. Though at first sight it may appear cumbrous and difficult, it is really nothing but the exact statement of a tacit assumption which we make every time that we estimate the length of a road drawn on a map, and treat successive small portions of it as if they were straight, adding them up by means of a ruler or the straight edge of a sheet of paper.

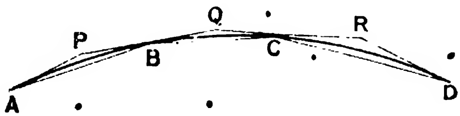


Fig. 70.

For we assume that the smaller the successive portions into which we divide the curved line, the more accurate will our estimate of its length become.

The postulate may be stated, for all curves here considered, as follows:—

Post. If a number of points be taken on any such curve and joined consecutively by straight lines, the perimeter of the open polygon so formed approaches a definite limit when the number of points is increased and the distance between every pair of consecutive points is decreased indefinitely; further, if an open polygon be formed by drawing the tangents at the above points, its perimeter approaches the same limit as that of the inscribed polygon.

* A proof in the special case of the circle is given in Lamb's "Infinitesimal Calculus" (Art. 4).

Def. The limit which is approached by the perimeter of the above open polygon is defined as the **length** of the curve.

It may be observed that the length of a curved line, just like that of a straight line, cannot be determined with perfect accuracy by measurement; it is only by calculation that the length of an (ideal) curve can be determined exactly.

PROP. 1. **The circumference of a circle bears a fixed ratio to its radius.**

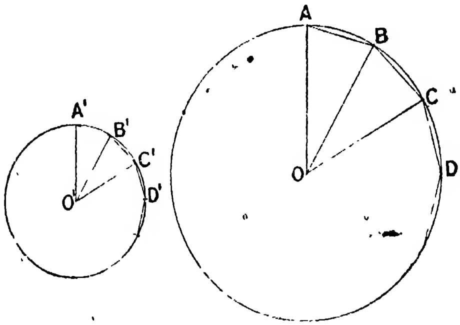


FIG. 71.

Let O, O′ be the centres of any two circles whatever, and let A, B, C be vertices of a polygon inscribed within the first circle. Draw O′A′, O′B′, O′C′ . . . parallel to OA, OB, OC . . . respectively.

In the triangles OAB, O′A′B′,

$$\frac{O'A'}{OA}=\frac{O'B'}{OB}, \text{ since } O'A'=O'B',\ OA=OB,$$

and the angles A′O′B′, AOB are equal.

The triangles are therefore similar. [Euc. vi. 6

Hence $\frac{A'B'}{AB} = \frac{O'A'}{OA}$, and similarly for all the other sides of the two polygons. Therefore the ratio of their perimeters is equal to that of the radii O′A′, OA.

But when the sides of the first polygon are all decreased indefinitely, these perimeters approach definite limits, which are the circumferences of the two circles.

That is, the circumferences are in the same ratio as the radii, which is the same thing as saying that the ratio of the circumference to the radius is constant.

Def. The constant ratio defined above cannot be expressed numerically with exactitude; that is, it is incommensurable. It is denoted by 2π, and the value of π is approximately

$$3{\cdot}14159\ \ldots$$

Approximations to the value of π can be found by calculating the perimeters of inscribed and circumscribed regular polygons containing 2^n or 3×2^n sides. Archimedes (287–212 B.C.), in his essay on the Measure of the Circle, by considering polygons of $96 (= 3 \times 2^5)$ sides, showed that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$.

35. The Measurement of Area.

When we state that the area of a geometrical figure is 356 square millimetres it is implied that, ideally at any rate, the figure could be subdivided into pieces which when put together again would make 356 actual squares, the side of each being one millimetre. So in all cases the fundamental idea of the area of a figure implies subdivision in exactly the same way as does the idea of the length of a straight line. But with respect to the actual subdivision in practice, there is a difference. It is not in general possible to subdivide a geometrical figure into a definite number of portions which can be fitted together again into square units of area. But in thought the mind passes on to the limit which is approached by the approximate subdivision into squares.

The proof of the existence of such a limit is not simple enough to be included here, and an additional postulate is therefore required. It is convenient to approach this limit by a readily applicable method; one in which an approximate measure of the area of a given figure is found by supposing it to be drawn upon squared paper. It is impossible to find the nearest whole number of elementary units of area contained in the given figure, because the smaller the size of the square unit the more incomplete squares will be contained in the figure. The postulate states that the total

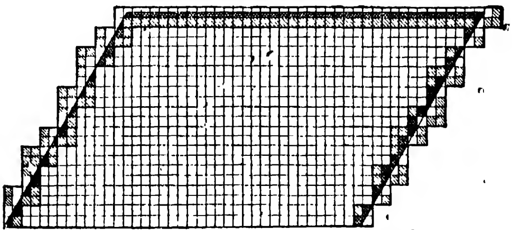


FIG. 72.

area of these incomplete squares decreases as their number increases, and that therefore the total number of complete squares gives an approximate measure of the required area. In the accompanying illustration the incomplete squares are shaded for two sets of squares, the linear dimensions of one set being double those of the other set. The smaller set of incomplete squares have the portions inside the figure doubly shaded, and the external portions unshaded to distinguish them from the larger set.

Post. If any plane figure be divided into a number of equal squares by two systems of parallel straight lines the total area * of the incomplete squares included within the figure can be made as small as we please, and that of the complete squares approaches a definite limit as their number increases and their size diminishes.

* It is assumed that these squares are submultiples of the standard unit.

Def. The limit which is approached by the area of the complete squares enclosed within the given figure when their size is diminished is the **area** of that figure.

The chief application of the preceding postulate is indicated by the first of the following scholiums:—

Scholium (*a*): An approximate measure of the area required can also be obtained by counting all the complete squares within the figure and as many of the incomplete squares as is convenient; and any such approximate measure approaches the area required as its limit.

For the importance of the incomplete squares taken all together can be made as small as we please by increasing their number.

Scholium (*b*): The area of a parallelogram is measured by the product

(length of base) × (height).

Divide the parallelogram into small squares which lie in rows parallel to the base.

An approximate measure of the area is obtained by multiplying together the number of complete units in the base and the number of complete rows contained by the height of the parallelogram. The limit approached by this measure when the unit is decreased is

(length of base) × (height).

Scholium (*c*): The area of the orthogonal projection of a plane figure of area S on a plane which is inclined to it at an angle θ is equal to S cos θ.

Divide the area into strips by straight lines perpendicular to the join of the two planes. The projection of each strip is of the same width as the strip, the area of the strip is therefore diminished in the ratio of the lengths of the strips, which is, by definition, equal to cos θ. But if the area of

each strip is diminished by projection in the same ratio, the whole area is diminished in that ratio.

But there is an alternative method of dealing with the areas of plane figures, and it is the method adopted in Euclid's Elements. In this treatment no explicit definition of area is given, but it is assumed that

(i) every figure has a definite area and congruent figures have the same area;

(ii) if equal areas be added to or subtracted from equal areas the resulting figures are of equal area, (though not necessarily congruent);

(iii) figures whose areas are the same submultiple of equal areas are of equal area (*e.g.* the halves of figures of equal area are equal).

By using these principles a square or rectangle may be found equal in area to any plane rectilinear figure. The method may be extended to curvilinear figures by means of the following postulate.

* **Post.** The area of a polygon inscribed or circumscribed to a closed curve approaches the area of the curve as a limit, when all the sides are diminished indefinitely.

† PROP. 2. **The area of a circle is measured by the product π (radius)^2.**

Let A, B, C . . . be the points of contact of the sides PQ, QR . . . of a polygon circumscribed to a circle whose centre is at O. The area of the polygon is the sum of the areas of the triangles OPQ, OQR . . . and is therefore equal to

$$\tfrac{1}{2}\,OA.PQ + \tfrac{1}{2}\,OB.QR + \ldots$$
$$= \tfrac{1}{2}\,.OA\,(PQ + QR + \ldots)$$
$$= \tfrac{1}{2}\,(\text{radius})\,(\text{perimeter of polygon}).$$

* This postulate is not independent of previous assumptions, and is therefore capable of proof.

† Euc. xii. 2 shows that the ratio area : (radius)^2 is constant.

When the length of each side diminishes, this approaches the definite limit

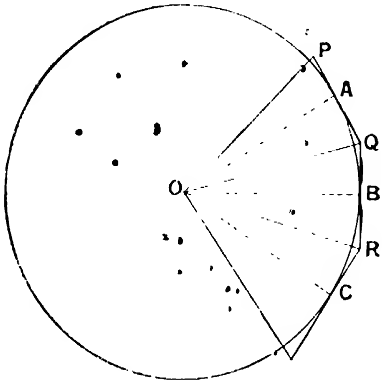


FIG. 73.

½ (radius) (circumference of circle) [Post., p. 107.
which is equal to π (radius)^2 [Def., p. 109.

36. The Measurement of Volume.

The measurement of volume may be treated in exactly the same way as the measurement of area, except that we imagine our figure drawn in a cubed space instead of on squared paper.

Post. If a solid figure be divided into a large number of cubes by three sets of parallel planes, the total volume* of the incomplete cubes included within the given figure can be made as small as we please, and that of the complete cubes approaches a definite limit as their number increases and their size diminishes.

* It is assumed that these cubes are submultiples of the standard unit.

Def. The limit approached by the volume of the complete cubes enclosed within the given figure, when their size is diminished, is the **volume** of that figure.

Scholium (*a*): An approximate measure of the volume required can also be obtained by counting all the complete cubes, and as many of the incomplete cubes as is convenient, and any such approximate measure approaches the required volume as a limit.

The preceding definitions may be applied directly to prove a general theorem concerning similar figures.

Def. **Similar figures** are such that all distances between points in one figure bear a constant ratio to the distances between the corresponding points of the other figure and that all corresponding angles are equal.

The existence of such figures may be shown to depend on Euc. vi. 4. and on Prop. 20, p. 48, since any polyhedron may be decomposed first into pyramids and then into tetrahedrons.

Scholium (*b*): The surface areas and volumes of similar figures vary respectively as the square and cube of their linear dimensions.

If, when the linear dimensions of a figure are altered, the elementary unit of length be altered in the same ratio, the surface area and volume of the figure are measured by the same number as before.

But the elementary units of area and volume have been altered respectively in the ratios of the square and cube of the ratio of the linear dimensions of the two figures.

Hence the surface area and volume of similar figures measured in terms of a standard unit are respectively proportional to the square and cube of its linear dimensions.

Examples XI.

1. The edges of a rectangular box arranged in descending order of magnitude are a, b, c respectively; find the shortest distance over the sides of the box from one corner to the

opposite corner, and the inclination of this path to the longest edges of the box.

2. The roof of a rectangular building, the base of which has dimensions 32′ by 16′, has four faces, each inclined to the horizontal at an angle of 45°; find the lengths of the five edges along which the faces join and their inclinations to the vertical. [16, 13·9, 54° 44′.

3. A room has dimensions 30, 12, and 12 ft.; a spider on the vertical medial line of one end-wall, and 3 ft. from the ceiling, wishes to reach a fly on the opposite end, 3 ft. from the ground and 6 ft. from either side-wall; find the shortest path along walls or ceiling. [41·78 ft.

4. What angle does a diagonal of a cube make with any face? [35° 16′.

5. Find the dihedral angles of a regular tetrahedron, and also the length of the normal from a vertex to opposite face, in terms of the length of the edge. [70° 32′; ·816.

6. The dimensions of a room are 21, 30, and 10 ft. respectively; find the inclination to the horizontal of any diagonal, and the angles which the vertical plane containing it makes with the walls. [15° 16′; 35°, 55°.

7. A roof makes an angle of 20° with the horizontal, and one of its edges makes an angle of 30° with its horizontal edge; what angle does that edge make with the horizontal? [9° 51′.

8. The dimensions of a room are 25, 20, and 12 ft.; find the magnitude and position of the shortest straight line joining a diagonal to either of the edges of the ceiling which it does not intersect. [10·8, 10·3 ft.; the diagonal is divided in the ratios ·83, ·36.

9. A pyramid stands symmetrically upon a square base, and the inclination of the faces to the base of the pyramid is 24°. Prove that the angle which an edge makes with the base is 17° 30′, and that the face angles at the vertex are 84° 50′.

10. Show that a square may be folded so as to form a tetrahedron with three concurrent edges mutually perpendicular. Calculate the least dihedral angle between its faces and also the smallest plane angle of any face.

[$\cos^{-1}\frac{2}{3} = 48° \; 12'$, $\tan^{-1}\frac{1}{2} = 26° \; 34'$.

11. Find the dihedral angle between two consecutive faces of a regular octahedron, and also the length of a diagonal in terms of the length of a side. [$109° \; 28'$; $\sqrt{2}$.

12. Find the dihedral angle between consecutive faces of a regular dodecahedron. [$63° \; 24'$.

*13. If two regular solids are conjugate, the ratio of the radius of the circumscribed sphere to that of the inscribed sphere is the same for both.

14. A plane makes equal angles with two vertical walls which are at right angles; show that if it is inclined to the vertical at an angle α, the lines of section with the walls are inclined to the vertical at an angle whose tangent is

$$\sqrt{2} \tan \alpha.$$

15. The traces of a straight line on two perpendicular planes are A and B, if A′, B′ be the orthogonal projections of A, B on the join of these planes, and

$$AA' = a, BB' = b, A'B' = c,$$

show that

$$AB = (a^2 + b^2 + c^2)^{\frac{1}{2}},$$

and that the tangents of the angles made by AB with the two planes and their join are respectively

$$\frac{b}{(a^2 + c^2)^{\frac{1}{2}}}, \quad \frac{a}{(b^2 + c^2)^{\frac{1}{2}}}, \quad \frac{(a^2 + b^2)^{\frac{1}{2}}}{c}.$$

* This proposition includes Propositions 2, 4 of the first of two books on the Regular Solids which have been appended to Euclid's Elements, and which are now attributed to Hypsicles of Alexandria (*circa* 500 A.D. ?).

16. The traces of a plane on two planes at right angles make angles α, β with their join; show that the tangents of the angles which the plane makes with the two perpendicular planes are respectively

$$\frac{\tan \alpha}{\sin \beta} \quad \text{and} \quad \frac{\tan \beta}{\sin \alpha}.$$

17. AB, BC are horizontal lines denoting the tops of two walls of a house which intersect at right angles, and the slopes of the roof which meet the walls in AB, BC are inclined to the vertical at angles α, β; show that if BD, the line of intersection of these two slopes, makes an angle θ with the vertical,

$$\tan^2\theta = \tan^2\alpha + \tan^2\beta$$

18. ABCD is a tetrahedron, and the three faces meeting in D are equal isosceles triangles of vertical angle 2α; prove that the angle between two isosceles faces is equal to

$$2\sin^{-1}(\tfrac{1}{2}\sec\alpha).$$

19. Three edges of a tetrahedron meeting at O have the same length a, and make equal angles α with one another; prove that the distance of O from the opposite face is

$$a\sqrt{\left(\frac{1+2\cos\alpha}{3}\right)}.$$

20. If OA, OB, OC are three mutually perpendicular straight lines, and O′ is the orthogonal projection of O on the plane ABC, the areas of the triangles AO′B, BOA, ABC are in geometric progression.

21. The sum of the areas of three faces of a tetrahedron is greater than the area of the fourth face.

22. Calculate the angles made by the radii marking the hours in a sundial (i) when the face is horizontal, (ii) when the face is vertical and looks south.

[$\tan\phi_n =$ (i) $\sin l \tan \frac{1}{6}n\pi$, (ii) $\cos l \tan \frac{1}{6}n\pi$.

23. The plane, passing through the mid-points of a pair of opposite edges of a tetrahedron, for which the area of its section is a minimum, is parallel to the shortest distance between one of the other pairs of opposite edges.

[See Ex. 24, p. 30.

24. Determine the plane parallel to two opposite edges of a tetrahedron which cuts it, so that the area of its section is a maximum. *N.B.*—$x(a-x)$ is a maximum when $x=\frac{1}{2}a$.

25. The minimum area intercepted by a sphere of radius a on a plane which passes through a point within the sphere at a distance d from its centre is $\pi(a^2-d^2)$.

26. The area intercepted between two concentric spheres is constant for all planes which cut both.

27. If two planes intersect at right angles in a given straight line, the sum of the areas intercepted on them by a given sphere is constant.

28. Show that if the areas of sections of two fixed spheres by a plane are in the same ratio as the areas of sections through the two centres, this plane cuts the line of centres in one of two fixed points.

29. Determine the lengths of the edges of a rectangular parallelepiped, knowing that they are proportional to a, b, c, and that its volume is V.

30. $ABCD\ A'B'C'D'$ is a parallelepiped, any point is taken in the diagonal AC', and three planes drawn through it parallel respectively to the three faces meeting in A. Prove that each cuts off an equal volume.

31. Determine a plane which, drawn parallel to the base of a triangular pyramid, shall divide the volume in the ratio $m:n$.

32. Determine a plane drawn parallel to the base of a given tetrahedron to cut off a tetrahedron whose total area is half that of the given tetrahedron; what is the volume of the new tetrahedron?

33. Prove geometrically, from the volume of a cube, that if a and b be whole numbers

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

34. Give a plane construction to determine a straight line, whose ratio to a given straight line is equal to that of the volumes of two cubes, the edges of which are known.

Scholium: If A, B, C are any three points in space, the algebraic sum of the projections of AB, BC is equal to the projection on that line of AC; if A, B, C, D are any four points, the algebraic sum of the projections of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ is equal to the projection of AD.

35. Being given the orthogonal projections of a straight line on three mutually perpendicular axes, find its projection on another straight line which makes given angles with the three axes.

36. The angle between two straight lines which make angles α, α', β, β', γ, γ' respectively with three mutually perpendicular axes is

$$\cos^{-1}(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma').$$

37. The square on a straight line is equal to the sum of the squares on its orthogonal projections upon three mutually perpendicular axes.

38. The sum of the squares of the cosines of the angles which a straight line makes with three mutually perpendicular axes is unity.

39. The sum of the squares of the cosines of the angles made by any plane with three mutually perpendicular planes is unity.

40. If A, A′ are the areas of two figures in the same plane, and P, P′, Q, Q′, R, R′ are their projections on three mutually perpendicular planes,

$$A \cdot A' = P \cdot P' + Q \cdot Q' + R \cdot R'.$$

41. The square of any plane area is equal to the sum of the squares of the areas of its projections upon three mutually perpendicular planes.

In the following examples a, b, c and A, B, C denote either the face angles and dihedral angles of a trihedral solid angle, or the sides and angles of the spherical triangle formed by its intersection with a sphere of unit radius, having its vertex as centre.

42. If $C = 90°$, $\cos c = \cos a . \cos b$.

43. What is the error in calculating, as if it were flat, the hypotenuse of a right-angled triangle with two equal sides, each 50 miles long, drawn on the surface of the earth, taking the earth's radius to be 4000 miles? [2 yards.

44. If $C = 90°$, $\tan A = \tan a / \sin b$.

45. If $C = 90°$, $\sin A = \sin a / \sin c$.

CHAPTER VI

THE PRISM AND CYLINDER

37. The Lateral Area of a Prism or Cylinder.

Def. A **right prism** is one in which the lateral edges are normal to its ends; an **oblique prism** is one in which the lateral edges are not normal to its ends.

Def. A **right section** of a prism is a section by a plane which is normal to its lateral edges.

PROP. 3. **The lateral area of a prism is measured by (perimeter of right section) × (length of lateral edge).**

(See Fig. 74, p. 122.)

Let ABC . . . A′B′C′ . . . be a prism in which the lateral edges, produced if necessary, meet a right section in abc . . . Let s be the perimeter of the right section, so that

$$s = ab + bc + \ldots,$$

and let l denote the length of each of the edges AA′, BB′, CC′ . . .

The area of the face ABB′A′ is measured by the product AA′ . ab, [Sch. (b), p. 111.

and the total lateral area is therefore

$$\begin{aligned} &l \cdot ab + l \cdot bc + \ldots \\ &= l(ab + bc + \ldots) = l \cdot s \end{aligned}$$

or (perimeter of right section) × (length of lateral edge).

The difficulty in giving a general definition of the area of

a curved surface may be avoided by dealing only with those simple cases with which we are concerned.

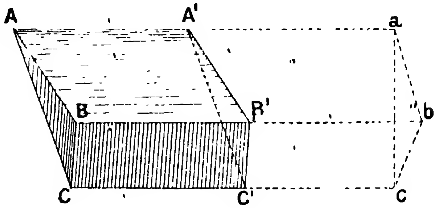


FIG. 74.

Def. A **cylindrical surface** is traced out by a straight line which is in a fixed direction and passes through a given curve; a **cylinder** is bounded by a cylindrical surface and two ends which are portions of parallel planes. Any position of the straight line generating the curved surface is a **generator**.

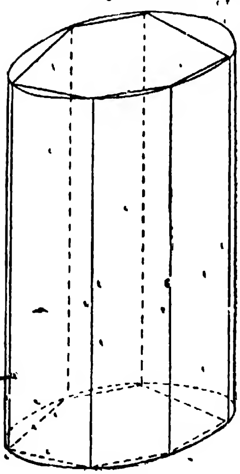


FIG. 75.

Def. Prisms, the lateral edges of which coincide with generators of a cylinder, are **inscribed** in that cylinder.

Def. The **area** of the curved surface of a cylinder is the limit approached by the lateral area of an inscribed prism when each side is indefinitely diminished in breadth.

That a definite limit is approached by the lateral area of a prism inscribed in a cylinder follows from the preceding proposition and the postulate of p. 107.

Scholium: The area of the curved surface of a cylinder is measured by

(circumference of right section) × (length of generator).

For the perimeter of the right-section of the prism is a polygon inscribed in the right-section of the cylinder by the same plane, and (Post., p. 107), when each side is indefinitely diminished, this perimeter approaches a definite limit, called the perimeter of the cylindrical cross-section.

38. The Volume of a Prism or Cylinder.

PROP. 4. **The volume of an oblique prism (or cylinder) is equal to that of a right prism (or cylinder), having the same right section and equal lateral edges.**

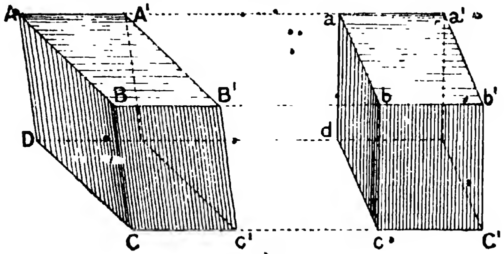


FIG. 76

Let ABCD . . A′B′C′D′ . . . be any oblique prism.

Consider the sides of the prism to be produced, and let abcd . . ., a b′c′d′ . . . be right sections such that aa′, bb′, cc′, are each equal to AA′.

It is required to prove that

Vol. ABCD . . . A′B′C′D′ = vol. abcd . . . a′b′c′d′ . . .

The truncated right prisms (or cylinders) abcd . . . ABCD . . . , a′b′c′d . . . A′B′C′D′ . . . are congruent, because their bases abcd . . . , a′b′c′d′ are congruent, and aA = a′A′, bB = b′B′ . . .

Taking these two equal volumes from the whole figure ABCD . . . a′b′c′d′ . . . , the remaining prisms ABCD . . . A′B′C′D′ . . . and abcd . . . a′b′c′d′ . . . are of equal volume.

Def. The **height** of a prism is the perpendicular distance between its two ends.

PROP. 5. **The volume of a prism (or cylinder) is measured by**

$$\text{(area of base)} \times \text{(height)} = \text{(area of right section)} \times \text{(length of lateral edge)}.$$

Suppose the prism divided into cubical elements of volume which lie in layers parallel to the ends. An approximate measure of the volume is obtained by multiplying together the number of complete squares contained in the base

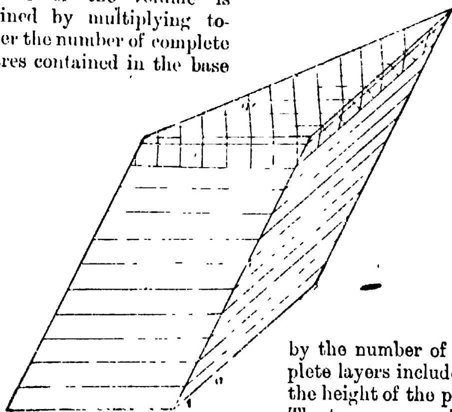


FIG. 77.

by the number of complete layers included in the height of the prism. The true measure is the limit approached by this when the cubes are diminished, and is therefore equal to

$$\text{(area of base)} \times \text{(height)}.$$

But it has already been shown (Prop. 4, p. 123) that the volume of an oblique prism (or cylinder) is equal to one of the same right section and of the same length of lateral edge.

Taking this right section as base, we obtain the alternative form,

$$\text{(area of right section)} \times \text{(length of lateral edge)}.$$

Scholium: The volume of a triangular prism is measured by half the product of the area of any side into its distance from the edge to which it is parallel. [Sch. (*b*), p. 111.

Examples XII.

1. In every triangular prism the area of the greatest lateral face is less than the sum of the two others.

2. The area of the section of the sides of a prism made by a plane is least when the plane is normal to the lateral edges of the prism.

3. The straight line joining the centroids of any two sections of a prism is parallel to its lateral edges.

Def. The centroid of a figure which can be divided into triangles of areas a_1, a_2, a_3 . . with centroids at P_1, P_2, P_3 . . respectively is found as follows: Divide P_1P_2 at P_{12} in the ratio $a_2 : a_1$, divide $P_{12}P_3$ at P_{123} in the ratio $a_3 : a_1 + a_2$, etc.

4. The floor of an attic is rectangular, being 14 ft. by 12 ft., the ceiling slopes from a height of 9 ft. along one of the longer sides to a height of 5 ft. along the opposite side; find its cubic capacity. [1176 cub. ft.

5. A rectangular sheet of paper 11 in. by 6 in. is curved so as to form the lateral surface of a right circular cylinder; find the volumes of the two cylinders so formed. [58, 31 cub. in.

6. A sheet of paper in the form of a parallelogram, whose sides are 12 in. and 5 in. and inclined to each other at an angle of 70°, is curved so as to form the lateral surface of a right circular cylinder; find the volumes of the two cylinders which can be formed. [54, 22 cub. in.

7. The ratio of the volumes of two right circular cylinders, of which the curved surfaces are equal in area, is equal to the ratio of the radii of their bases or the inverse ratio of their heights.

8. The ratio of the areas of the curved surfaces of two right circular cylinders which are of equal volume is equal to the ratio of the square roots of their heights or the inverse ratio of the radii of their bases.

9. The volumes generated by turning a rectangle about adjacent sides successively are a and b cubic cms. respectively; what is the length of the diagonal of the rectangle?

10. What is the ratio of the volumes generated by rotating a parallelogram about two adjacent sides?

CHAPTER VII

THE PYRAMID AND CONE

39. Introductory Propositions.

Def. A conical surface is generated by a straight line which meets a fixed point and a given curve; a cone is bounded by a conical surface and a plane base.

Prop. 6. If two pyramids (or cones) are on equal bases and are of the same height, they are of equal volume.

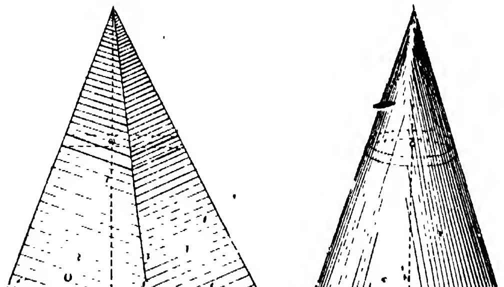


FIG. 78.

Suppose each pyramid (or cone) divided into slices all of equal thickness.

Sections of a pyramid (or cone) are similar to the base and of linear dimensions proportional to their distances from the vertex. [Prop. 24, p. 58.

Their areas are therefore proportional to the squares of their distances from the vertex. [Sch. (b), p. 114.

Hence the areas of sections of two pyramids at the same distance from their respective vertices bear the same ratio to the areas of the bases. These bases are equal in area, and the sections are therefore also of equal area.

Hence approximate measures of the volumes of corresponding slices are equal, and therefore also those of the total volumes. But the accurate measures are the limits approached by the approximate measures, and the exact volumes of the two pyramids (or cones) are therefore also equal.*

†PROP. 7. **The volume of a triangular pyramid is one-third that of a prism standing on the same base and of the same height.**

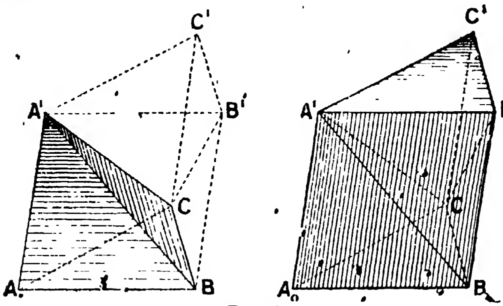


FIG. 79.

Let the triangular pyramid be denoted by ABCA′ and let ABC A′B′C′ be a prism on the same base ABC, having its lateral edges parallel to AA′. This prism is of the same height as the pyramid, and may be considered as made up of the three triangular pyramids C A′BB′, C AA′B, C A′B′C′.

The pyramids C A′BB′, C AA′B may be regarded as having

* Post. p. 113. † Euc. xii. 7.

a common vertex C, and as being on equal bases A′BB′, AA′B; their volumes are therefore equal. [Prop. 6, p. 126.

Similarly the pyramids C B′A′B, C A′B′C′ may be regarded as having a common vertex A′, and bases of equal area CBB′, CB′C′; their volumes are therefore equal. [Prop. 6, p. 126.

Thus the prism is made up of three triangular pyramids of equal volume. That is, the volume of ABC A′ is one-third that of a prism of the same height and on the same base.

40. The Volume of a Pyramid or Cone.

PROP. 8. **The volume of a pyramid (or cone) is measured by the product**

$$\frac{1}{3}\,(\textbf{area of base}) \times (\textbf{height}).$$

The volume of a triangular pyramid is one-third that of a prism on the same base and of the same height. [Prop. 7.

Its volume is therefore measured by

$$\frac{1}{3}\,(\text{area of base}) \times (\text{height}).$$

The volume of any other pyramid (or cone) is equal to that of a triangular pyramid on a base of equal area; [Prop. 6.

It also is therefore measured by

$$\frac{1}{3}\,(\text{area of base}) \times (\text{height}).$$

Examples XIII.

1. One corner of a parallelepiped is chipped off, so that the edges meeting in that vertex are reduced to $\frac{1}{2}$, $\frac{2}{3}$, $\frac{6}{7}$ of their former size respectively; the volume is reduced to $\frac{251}{252}$ of its former size.

2. The plane determined by one edge of a tetrahedron and the mid-point of the opposite edge divides the tetrahedron into two parts of equal volume.

3. A regular octahedron is formed whose corners are the middle points of the faces of a cube. Show that the volume of the octahedron is one-sixth the volume of the cube.

4. If a tetrahedron has one trihedral angle fixed, its volume is proportional to the product of the lengths of the three edges which meet in that point.

5. The volume of a tetrahedron formed by joining the centroids of the faces of a given tetrahedron is one twenty-seventh that of the given tetrahedron.

6. If a tetrahedron has one edge fixed as well as the dihedral angle which is formed there, its volume is proportional to the product of the areas of the two faces which meet in that edge.

7. The two tetrahedra inscribed in a given parallelepiped are of equal volume.

8. The volume of the octahedron common to two tetrahedra inscribed in the same parallelepiped is one-sixth that of the parallelepiped.

9. The volume of a tetrahedron ABCD remains fixed when C, D move in any way on given straight lines, parallel to AB, which is fixed.

10. The plane which bisects the dihedral angle between two faces of a tetrahedron divides the opposite edge in the ratio which the areas of the two faces bear to one another. Is the same statement true for the bisector of the exterior angle between the planes?

11. The volume of a tetrahedron is one-third of the volume of the circumscribed parallelepiped. [See Fig., p. 57.

12. Determine a point within a tetrahedron such that the planes joining it to the six edges divide the figure into four tetrahedra of equal volume.

13. If on two given straight lines constant lengths AB, CD are marked off in any positions, the volume of the tetrahedron ABCD is fixed.

14. If A, B, C, D denote the areas of the faces of any tetrahedron, and V its volume, prove that the radius of the inscribed sphere is

$$r = \frac{3V}{A+B+C+D}.$$

15. In a tetrahedron in which opposite edges are perpendicular the products of the lengths of pairs of opposite edges are inversely proportional to the shortest distances between them.

16. The sum of the normals drawn to the faces of a regular polyhedron from any internal point is constant.

17. Determine a plane passing through a given straight line in one face of a trihedral angle which shall cut off a volume equal to that cut off by a given plane.

18. If O be any point in a face BCD of a tetrahedron ABCD and straight lines drawn through O parallel to AB, AC, AD respectively meet the faces ACD, ADB, ABC in P, Q, R respectively

$$\frac{OP}{AB}+\frac{OQ}{AC}+\frac{OR}{AD}=1.$$

19. Every plane which bisects two opposite edges of a tetrahedron divides it into two parts of equal volume.

[See Ex. 25, p. 68.

20. If a point V is such that the volumes of pyramids with a common vertex at V and on given bases not co-planar have a given ratio, the locus of V is a pair of planes passing through the join of the planes of the two given bases. Extend this theorem to the case of three plane bases.

21. There is in general one point V such that the volumes of the pyramids with a common vertex at V and four given plane polygons of areas a_1, a_2, a_3, a_4 for bases have the ratio $n_1 : n_2 : n_3 : n_4$; the distances of this point from the planes of the four polygons are proportional to

$$\frac{n_1}{a_1} : \frac{n_2}{a_2} : \frac{n_3}{a_3} : \frac{n_4}{a_4}.$$

22. If O is any point within a tetrahedron ABCD and the line joining any vertex A and O meet the opposite face in a, then

$$\frac{Oa}{Aa}+\frac{Ob}{Bb}+\frac{Oc}{Cc}+\frac{Od}{Dd}=1.$$

23. Determine a plane drawn parallel to a given straight line or passing through a given point which shall divide a given tetrahedron into two equal volumes.

24. Given the four heights h_1, h_2, h_3, h_4 of a tetrahedron and the distances p_1, p_2, p_3 of a point from three of its faces, the distance p_4 of this point from the fourth face is given by

$$p_4=h_4\left(1-\frac{p_1}{h_1}-\frac{p_2}{h_2}-\frac{p_3}{h_3}\right)$$

25. A convex polyhedron has n faces and the r^{th} face is of area A_r; p_r denotes the distance of a point P from the plane of the r^{th} face, to be measured positive or negative as the point is on the same side of the plane as the polyhedron or the reverse; then is $\sum_{r=1}^{r=n} A_r p_r$ constant for all positions of P, inside or outside.

26. If the straight line through A equally inclined to the three faces of a tetrahedron ABCD, which contain A, meets the opposite face in A′, the areas of the triangles A′BC, A′CD, A′DB are proportional respectively to the areas of the faces ABC, ACD, ADB.

27. If in the face BCD of a tetrahedron a point A′ is taken so that the areas of the triangles A′CD, A′BD, A′BC are proportional to numbers y, z, w and similarly for the other faces (numbers x, y, z, w being associated with the points A, B, C, D respectively), AA′, BB′, CC′, DD′ are concurrent in a point O, such that the volumes OBCD, OCDA, ODAB and OABC are proportional to x, y, z and w respectively.

41. The Volume of a Frustum of a Pyramid (or Cone); Simpson's Rule.

Def. A frustum of a pyramid (or cone) is bounded by its base and a plane section which is parallel to it.

Prop. 9. The volume of a frustum of a pyramid (or cone) is measured by

$$\tfrac{1}{3}h(X+Y+\sqrt{XY})=\tfrac{1}{6}h(X+Y+4M),$$

where h is the height, X, Y are the areas of its ends, and M is the area of a section midway between them.

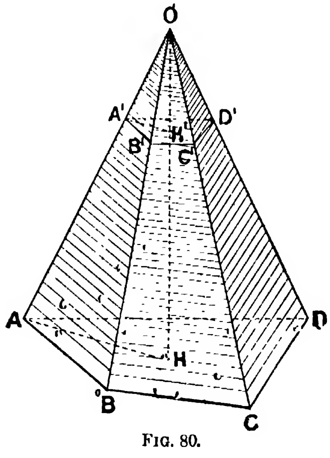


Fig. 80.

Let O be the vertex of the pyramid whose frustum is bounded by the ends $ABCD$ . . ., $A'B'C'D'$ . . .; and let the normal to these faces from the vertex cut them in points H, H' respectively

Denote the required volume by V, and the lengths OH, OH' by p, q respectively.

Then $\quad V = \text{vol. } OABCD \ldots - \text{vol. } OA'B'C'D' \ldots,$

and $$h = p - q \quad \ldots \ldots \quad (1)$$

Hence $$V = \frac{1}{3}(Xp - Yq) \quad . \text{ [Prop. 8, p. 128.} \quad (2)$$

But the areas of the faces $ABCD \ldots$, $A'B'C'D' \ldots$ are proportional to the squares of their distances from the vertex. [Prop. 24, p. 58; Euc. vi.

That is $$\frac{X}{p^2} = \frac{Y}{q^2} = x, \text{ say} \quad \ldots \ldots \quad (3)$$

Substitute now in equation (2), and

$$V = \frac{1}{3}x(p^3 - q^3),$$

$$= \frac{1}{3}x(p - q)(p^2 + pq + q^2). \quad \ldots \ldots \quad (4)$$

Making use of equation (1),

$$V = \frac{h}{3} \cdot (xp^2 + xpq + xq^2). \quad \ldots \ldots \quad (5)$$

Then, making use of equations (3),

since $\quad xpq = \sqrt{xp^2 \cdot xq^2} = \sqrt{XY},$

we have $$V = \frac{1}{3}h(X + \sqrt{XY} + Y) \quad \ldots \ldots \quad (6)$$

Secondly, the length of the normal from the vertex to the mid-section is $\quad \frac{1}{2}(p + q),$

and $\therefore$ $$\frac{4M}{(p+q)^2} = \frac{X}{p^2} = \frac{Y}{q^2} = x \quad \ldots \ldots \quad (7)$$

$$\text{Hence } \frac{1}{6}h(X+Y+4M)=\frac{1}{6}h\cdot x\{p^2+q^2+(p+q)^2\}$$

$$=\frac{1}{6}hx\{2p^2+2q^2+2pq\}$$

$$=\frac{1}{3}h\{xp^2+xq^2+xpq\},$$

which by equation (5) is equal to V.

$$\text{Hence}\quad V=\frac{1}{3}h(X+Y+\sqrt{XY})=\frac{1}{6}h(X+Y+4M).$$

The last formula expresses what is known as Simpson's* Rule; it is a rule of wide application, as is shown in books on the Integral Calculus. In particular it may be extended at once to any polyhedron in which all the vertices lie in two parallel faces, whether these faces contain the same number of edges or not. The steps of the proof are given in the Scholiums which follow.

Def. Any polyhedron in which all the vertices lie in two parallel faces is a **prismatoid.**

Scholium (*a*): Simpson's Rule may be applied to a complete pyramid or cone.

For this is the limiting case of a frustum.

Scholium (*b*): Simpson's Rule may be applied to a triangular prism in which one lateral edge and the opposite lateral face are treated as two parallel faces.

For $X=0$, $M=\frac{1}{4}Y$, and therefore $V=\frac{1}{2}hY$, which agrees with Scholium, p. 121.

Scholium (*c*): Simpson's Rule may be applied to the sum or difference of two figures of the same height for which it is

* Thomas Simpson (1710–1761).

known to be true, and which lie between the same two parallel planes.

For if V_1, X_1, Y_1, M_1 and V_2, X_2, Y_2, M_2 refer to the two given figures, and V', X', Y', M' refer to the figure which is their sum or difference,

$$V' = V_1 \pm V_2,\ X' = X_1 \pm X_2,\ Y' = Y_1 \pm Y_2,\ M' = M_1 \pm M_2.$$

Hence $$V' = \tfrac{1}{6}h(X_1 + Y_1 + 4M_1) \pm \tfrac{1}{6}h(X_2 + Y_2 + 4M_2)$$
$$= \tfrac{1}{6}h(X' + Y' + 4M').$$

That is, if Simpson's Rule holds for two figures of the same height taken separately, it may be applied when they are taken together.

Scholium (a): Simpson's Rule may be applied to a tetrahedron if two opposite edges are regarded as parallel faces.

For a tetrahedron may be regarded as the difference between a triangular prism and a quadrangular pyramid. (See figure on p. 127.)

Scholium (b): Simpson's Rule may be applied to any prismatoid.

For, by joining any vertex to all the other vertices, the prismatoid can be divided into pyramids and tetrahedra all lying between the same two parallel planes. Simpson's Rule holds when they are taken separately, and therefore also when they are taken together.

42. The Volume of a Truncated Triangular Prism.

Def. A **truncated prism** is the portion of a prism cut off by a plane which is not parallel to the ends of the prism and which meets all its lateral edges.

PROP. 10. **The volume of a truncated triangular prism is measured by**

$$\frac{1}{3}(\text{area of right section}) \times (\text{sum of lateral edges}).$$

Let ABC A′B′C′ be a truncated triangular prism.

Let the parallel edges A′A, B′B, C′C be produced to meet a right section in a, b, c respectively, and let the area of this section be denoted by S.

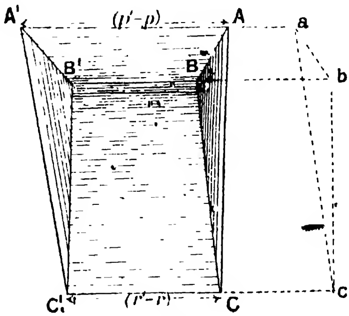


FIG. 81.

Let the lengths Aa, A′a, Bb, . . . be denoted by p, p′, q, q′, r, r′ respectively, and the volume required by V, so that

$$V = \text{vol. } A'B'C'abc - \text{vol. } ABCabc.$$

The lengths of the lateral edges are $(p'-p), (q'-q), (r'-r)$.

Just as in Prop. 7, p. 127, a triangular prism is divided into three equal triangular pyramids, the truncated prism ABCabc may be divided into three unequal pyramids,

CABb, CAab, Cabc.

The volume of the last of these, since Cc is normal to its base abc, is $\frac{1}{3}$r S.

Because Cc is parallel to the face AaL,

$$\text{vol. CAab} = \text{vol. cAab}. \quad \text{[Prop. 8, p. 128.}$$

But Aa is normal to the face abc,

whence $$\text{vol. CAab} = \frac{1}{3}\text{pS}.$$

Similarly,

$$\text{vol. CABb} = \text{vol. cABb} = \text{vol caBb}.$$

whence $$\text{vol. CABb} = \frac{1}{3}\text{qS}.$$

Hence $$\text{vol. ABCabc} = \frac{1}{3}(\text{p} + \text{q} + \text{r})\text{S}.$$

Similarly $$\text{vol. A}'\text{B}'\text{C}'\text{abc} = \frac{1}{3}(\text{p}' + \text{q}' + \text{r}')\text{S},$$

and $$\text{V} = \frac{1}{3}(\text{p}' - \text{p} + \text{q}' - \text{q} + \text{r}' - \text{r})\text{S}$$

$$= \frac{1}{3}(\text{area of right section}) \times (\text{sum of lateral edges}).$$

43. The Regular Pyramid and Right Circular Cone.

Def. A regular pyramid is one whose lateral faces are congruent isosceles triangles; from which it follows that the base is a regular polygon.

Def. A pyramid whose lateral edges coincide with generators of a cone is **inscribed** in that cone.

PROP. 11. **A regular pyramid can be inscribed in a right circular cone.**

Let OABC . . . be a regular pyramid and let K be the projection upon the base of the vertex O.

The right-angled triangles OKA, OKB, OKC . . ., have

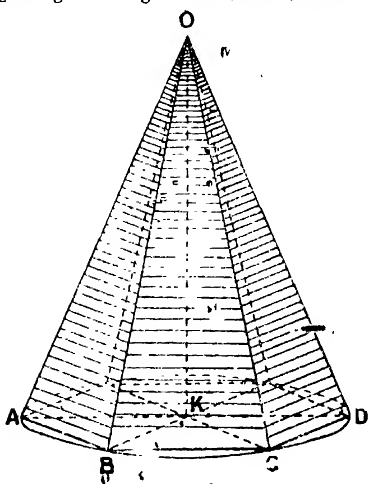


FIG. 82.

OA = OB = OC = . . . and OK common, and are therefore congruent.

Hence OA, OB, OC . . . are all generators of the right circular cone generated by rotating the right-angled triangle OAK about OK as axis.

The pyramid is therefore inscribed within this cone.

Def. The **slant height** of the frustum of a regular pyramid is the perpendicular distance between parallel edges of its

lateral faces; that of the frustum of a right circular cone is the length of a generator.

Def. The **mid-section** of a frustum of a pyramid or of a cone is a section parallel to its ends and half-way between them.

PROP. 12. The area of the lateral surface of a frustum of a regular pyramid is measured by

(perimeter of mid-section) × (slant height).

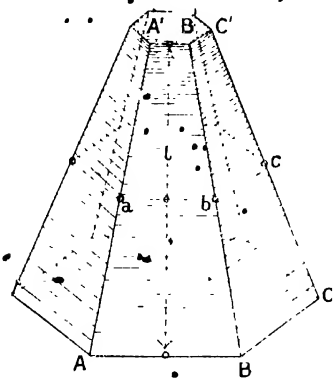


FIG. 83.

Let the frustum be denoted by ABC... A'B'C'... and the mid-section by abc ... Let the slant height be l.

The area of the face AA'B'B is equal to that of the parallelogram formed by drawing a parallel to AA' through b, the mid-point of BB'. [Euc. I. 26.

It is therefore measured by ab . l, and the total lateral surface is n . ab . l, where n is the number of lateral faces.

But n . ab is the perimeter of the mid-section, and the total area is therefore measured by

(perimeter of mid-section) × (slant height).

Scholium (*a*): The lateral surface of a regular pyramid is measured by the same formula.

Def. The **area of the curved surface of a cone** is the limit approached by the lateral area of an inscribed pyramid, when the breadth of each lateral face is indefinitely diminished.

That the lateral area of the inscribed pyramid does approach a definite limit is proved in the following scholium.

Scholium (*b*): The curved surface of a right circular cone or the frustum of one is measured by

(circumference of mid-section) × (slant height).

For the mid-section of an inscribed pyramid is a polygon inscribed within the mid-section of the circumscribing cone, and its perimeter therefore approaches the circumference as its limit. [Post., p. 107.

Examples XIV.

1. Prove the formula for the volume of the frustum of a triangular pyramid by splitting it up into three tetrahedrons.

2. Assuming the formula for the volume of the frustum of a triangular pyramid, deduce that of the frustum of any other pyramid.

3. Two opposite faces of a cube, ABCD and A′B′C′D′ are cut by a plane meeting AB, AD in H, K, and A′B′, A′D′ in H′, K′. Express the volume cut off in terms of the lengths of AH, AK, A′H′ and AA′.

4. The volume of a frustum of a pyramid of a height h with end faces of areas X, Y fall short of that of a prism of the same height but on a base of area $\frac{1}{2}(X+Y)$ by $\frac{h}{6}(\sqrt{X}-\sqrt{Y})^2$; the volume exceeds the prism of the same height on its mid-section as base by half this amount, that is, by

$$\frac{h}{12}(\sqrt{X}-\sqrt{Y})^2.$$

5. The volume of a tetrahedron is measured by two-thirds the product of the shortest distance between two opposite edges and the area of the parallelogram formed by joining the centres of the four remaining edges.

6. Verify that the volume of a truncated triangular prism is given by Simpson's Rule.

7. A plane divides AA′, BB′, CC′, DD′, parallel edges of a rectangular block, in the ratios 1 : 6, 2 : 5, 5 : 2, and 4 : 3 respectively; show that it divides the block into volumes which are in the ratio 3 : 4.

8. Through any given straight line one plane can be drawn dividing a parallelepiped into two parts of equal volume.

9. If along four parallel edges of a parallelepiped distances a, b, c, d are measured, the volume of the tetrahedron of which these four points are vertices is proportional to (a − b + c − d).

10. From two parallel edges of a triangular prism a plane which divides the volume into two equal parts cuts off lengths a, b; what length is cut off from the third parallel edge?

11. A rhombus revolves about a diagonal: the area of the surface described is equal to that of a circle whose radius is a mean proportional between the side of the rhombus and the other diagonal.

12. If a plane drawn through one edge of the base of a regular quadrangular pyramid divides it into two equal volumes, it must divide the two lateral edges opposite to the given edge in medial section. [·62.

13. A regular pyramid has a triangular base, each side of which is equal to a; if the ratio of the area of total surface : volume be p : a, then the height is $6pa/(p^2 - 108)$.

14. The volume of a truncated triangular prism is equal to the product of the area of its right section into the distance between the centroids of its end faces. Can this theorem be extended to any truncated prism? [See Ex. ?, p. 125.

15. A plane which cuts off a constant volume from a triangular prism, passes through a fixed point.

16. The volume of a frustum of a right circular cone is to be 200 cubic feet, its height is to be 10 feet, and the radius of its base to be 3 feet; what must be the radius of the opposite face? [2·00 ft.

17. What must be the ratio of the radii of the end faces of a frustum of a right circular cone in order that its volume may be half that of the cylinder of the same height on the same base? [·366.

18. Calculate the area of curved surface, the total area, and the volume of a cone of which the meridian section is an equilateral triangle, in terms of one side of this triangle; what is the length of this line when the total area is 1 square foot and when the volume is 1 cubic foot? [·65; 1·64.

19. A symmetrical heap of stones has for base a rectangle 30 feet by 7 feet, and a level top 26 feet by 5 feet, and its height is 2 feet; find the volume. [$337\frac{1}{3}$ cub. ft.

20. The volume of a truncated quadrangular prism of which two opposite lateral faces are parallel and rectangles of sides a, b and c, d respectively, the distance between them being h, is measured by

$$\tfrac{1}{6}h\{(a+c)(b+d)+ab+cd\}.$$

21. The whole surface of a right circular cone is equal to that of a circle, the diameter of which is a mean proportional between the base and perimeter of any meridian section.

22. If the curved surface of a frustum of a right circular cone has an area equal to twice the difference between the areas of the circular ends, the semi-vertical angle of the cone is 30°.

23. Divide the lateral area of a right circular cone into n equal parts by planes parallel to the base.

24. The curved surface of a right circular cone develops into a circular sector of angle a; show that the semi-vertical angle of the cone is $\sin^{-1}(a/2\pi)$.

25. The volume of a right circular cone is one-third of the product of the area of its whole surface and the radius of the inscribed sphere.

26. The curved surface of a solid right circular cone, of semi-vertical angle α, and altitude h, rolls on a horizontal plane, the vertex being fixed. Find the area of the curved surface of the cone generated by the axis. [$\pi h^2 \cos \alpha$.]

27. A cylinder is inscribed in a right circular cone, and its height is one-half that of the cone; prove that its volume is three-eighths that of the cone. Also find the ratio of the curved surfaces. [$\frac{1}{2} \cos \alpha$.

28. The volumes of the figures produced by rotating a regular hexagon about the lines joining (i) opposite vertices, (ii) mid-points of opposite sides, are in the ratio ·99.

29. If AB is a generator of a frustum of a right circular cone and $A'B'$ is its axis, and OP bisects AB at right angles, cutting $A'B'$ in O, the area of its curved surface is measured by $2\pi . A'B' . OP$.

30. The surface areas of the figures produced by rotating a regular hexagon about the lines joining (i) opposite vertices, (ii) mid-points of opposite sides, are in the ratio ·99.

CHAPTER VIII

THE SPHERE

44. The Area of a Spherical Zone.

Def. A zone of a spherical surface is the portion intercepted between two parallel planes.

Scholium: A zone of a sphere may be supposed to be generated by rotating an arc of a circle about a diameter.

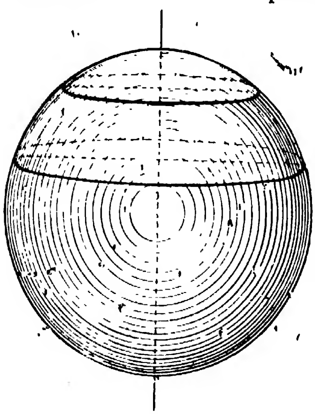


FIG. 81.

Def. If a surface of revolution be produced by rotating those sides of a regular polygon circumscribed to a circle which touch a given arc, about some diameter, the limit which the area of its curved surface approaches when the number of sides of the polygon is indefinitely increased is the area of the spherical zone generated by rotating the given arc.

That such a limit exists is proved in the following proposition.

* PROP. 13. **The area of any zone of a sphere is measured by**

2π (radius of sphere) $\times$ (thickness of zone).

Let O be the centre of the sphere, and let the normal drawn through O to the planes, which determine the zone, meet them in a, b, and let any plane through ab meet them in two chords AaA′, BbB′.

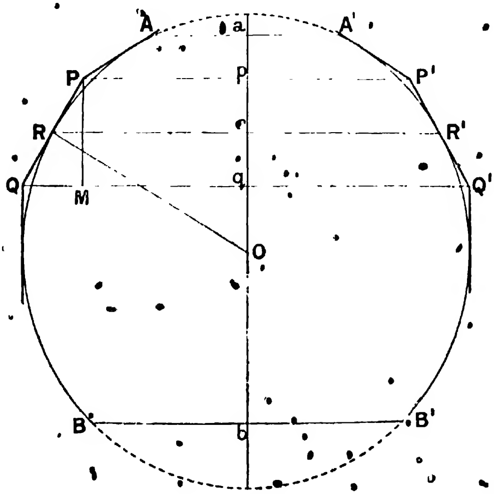


FIG. 85.

The zone may be considered as produced by rotating the figure aABb about ab, and its area is the limit approached by the area of the surface of revolution which is produced by rotating the sides of a regular polygon circumscribed to the arc AB.

* This theorem is proved by Archimedes (287–212 B.C.) in "The Sphere and Cylinder," Bk. i. Prop. 33.

Let PQ be any one of the sides of this polygon; if R is its point of contact, R is, by symmetry, the mid-point of PQ. Let p, q, r be projections on the axis ab of P, Q, R respectively; and let PM be drawn parallel to ab, meeting Qq in M.

The side PQ traces out the frustum of a conical surface, the area of which is measured by

$$2\pi \,.\, Rr \,.\, PQ, \qquad \text{[Sch. (}b\text{), p. 140.}$$

$2\pi \,.\, Rr$ measuring the perimeter of its mid-section.

But the triangles ORr, PQM are similar, because each side of one is perpendicular to the corresponding side of the other.

Hence $$\frac{OR}{Rr} = \frac{PQ}{PM}; \qquad \text{[Euc. vi. 4.}$$

but $PM = pq$, whence

$$Rr \,.\, PQ = OR \,.\, pq;$$

and the area traced out by PQ is therefore given by

$$2\pi \,.\, OR \,.\, pq.$$

By adding the areas traced out by each side of the polygon we obtain

$$2\pi \,.\, OR \,.\, (\text{projection on ab of the perimeter}).$$

But by making the sides of the polygon smaller and smaller we can make the projection on ab of the perimeter differ from ab by as little as we please.

The limit approached by this area, that is, the area of the zone, is therefore

$$2\pi \,.\, OR \,.\, ab,$$

which is

$$2\pi\,(\text{radius of sphere}) \,.\, (\text{thickness of zone}).$$

Scholium (*a*): The area of the whole spherical surface is $4\pi a^2$ where a is the radius of the sphere.

Def. A **spherical cap** is the portion of a spherical surface cut off by any plane; its **pole** is the point in which it meets the diameter normal to the plane by which it is cut off.

Scholium (*b*): The area of a spherical cap of height h, on a sphere of radius a, is $2\pi ah$.

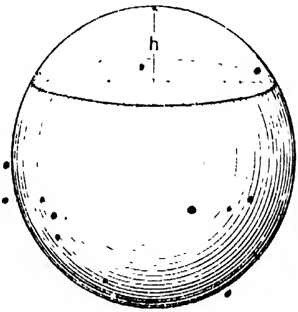


FIG. 86.

Having now obtained a meaning for the area of any zone of a sphere, we have also given a meaning to the area of any portion of a spherical surface which can be formed by the addition of zones or their subdivision into congruent portions. By this sort of process we can now give a meaning to the area of any portion of a spherical surface; the area of any spherical triangle is dealt with in the concluding section.

45. The Volume of a Sphere.

Def. A polyhedron is **circumscribed** to a surface when every surface is a tangent plane.

Post. The area of any portion of a spherical surface is the limit approached by the area of a circumscribed polyhedron when every face is indefinitely diminished, and the volume cut off by the spherical surface is the limit of that cut off by the polyhedral surface.

PROP. 14. **The volume of a solid which is bounded by a portion of spherical surface, of area A, and the conical surface joining its boundary to the centre of the sphere, which is of radius a, is**

$$\frac{1}{3}a \cdot A.$$

Suppose a polyhedral surface with a large number of faces to be circumscribed to the surface of area A; the volume required is the limit approached by the sum of the volumes of the pyramids on these faces as bases, with a common vertex at the centre. They are all of the same height.

[Prop. 28, p. 30.

Therefore their volume is measured by

$$\frac{1}{3}a \text{ (area of polyhedral surface).}$$

But the limit approached by this area is the area of the spherical surface which is A, and therefore the volume of the solid bounded by the spherical surface and the cone is

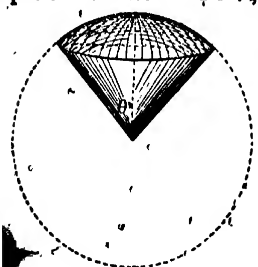


FIG. 87.

$$\frac{1}{3}aA.$$

*Scholium (*a*): The volume of the whole sphere is

$$\frac{4}{3}\pi a^3.$$

Def. A **spherical sector** is the portion of a sphere included within a right circular cone with its vertex at the centre.

Scholium (*b*): The volume of a spherical sector of semi-angle θ is

$$\frac{2}{3}\pi a^3(1 - \cos\theta).$$

* Euc. xii. 18 shows that the ratio volume : a^3 is constant.

be regarded

46. The Volume of a Spherical Segment.

Def. A segment of a sphere is a portion cut off by any plane; it is bounded by a spherical cap and a plane section called its base.

Def. A frustum of a sphere is a portion cut off between two parallel planes.

Prop. 15. The volume of the segment of a sphere is measured by

$$\pi h^2(a-\frac{1}{3}h)=\frac{1}{6}\pi h(h^2+3r^2),$$

where a = the radius of the sphere, h = the height of the segment, r = radius of the base.

Let the diameter HK drawn through O, the centre of the sphere, normal to the base of the segment, meet it in C. Let AB be any diameter of the base so that we have

$$OH=a,\ CH=h,$$
$$CA=CB=r.$$

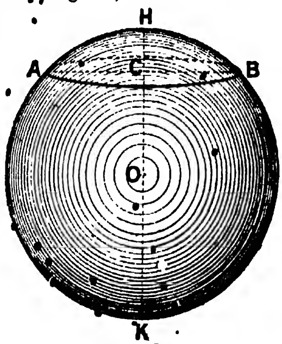


FIG. 88.

The volume of the segment is the difference between the volume of the spherical sector $OAHB$ and the cone OAB. It is therefore measured by

$$\frac{1}{3}a2\pi ah-\frac{1}{3}OC\cdot\pi r^2.$$

But $OC=a-h$, whence, denoting the volume by V,

$$V=\frac{1}{3}\pi(2a^2h-(a-h)r^2). \qquad (1)$$

From this result we can eliminate either of the quantities h, r, a, by making use of the equation

$$AC^2=CK\cdot CH. \qquad \text{[Euc. iii. 35}$$

Prop. 14 be written
a portion

$$\left.\begin{array}{r} r^2 = h(2a - h), \\ \text{or} \quad h^2 + r^2 = 2ah. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating r we have from equations (1) and (2),

$$V = \frac{1}{3}\pi\{2a^2h - h(a - h)(2a - h)\}$$

$$= \frac{1}{3}\pi h^2(3a - h).$$

That is,

$$V = \pi h^2(a - \frac{1}{3}h). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eliminating a, we have from equations (2) and (3)

$$V = \pi h\left(\frac{1}{2}h^2 + \frac{1}{2}r^2 - \frac{1}{3}h^2\right).$$

Which gives

$$V = \frac{1}{6}\pi h(h^2 + 3r^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

47. The Area of a Spherical Triangle.

Def. Any one of the four portions into which a spherical surface is divided by two diametral planes is a lune; the angle of the lune is the angle between the pair of half-planes which bound it.

Prop. 16. The area of a lune of a sphere is measured by $2\theta a^2$, where θ is the circular measure of the angle, and a is the radius of the sphere.

Two lunes of equal angle on the same sphere are congruent, and therefore of equal area.

By subdividing both lunes into a large number of equal lunes, an exact or approximate measure of the ratios of the angles or areas of the two lunes can be found; and these ratios are necessarily equal.

Hence, since the whole spherical surface may be regarded as a lune of angle 2π, we obtain

$$\frac{\text{area of lune}}{\text{area of sphere}} = \frac{\theta}{2\pi}.$$

But the area of the sphere is $4\pi a^2$; [Sch. (*a*), p. 147.
and hence the

$$\text{area of lune} = 2\theta a^2.$$

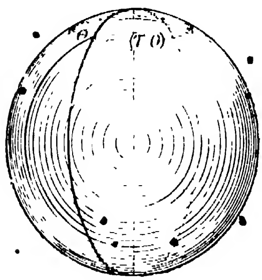


FIG. 89.

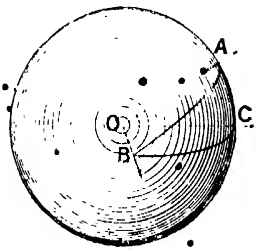


FIG. 90.

Def. A **spherical triangle** is the figure formed by the intersection with a spherical surface of a trihedral angle which has its vertex at the centre of the sphere, the angles of a spherical triangle are the dihedral angles of the trihedral solid angle.

Def. The sum of the measures of the angles of a spherical triangle less the measure of two right angles is called the **spherical excess** of the triangle.

In the triangle ABC the spherical excess is

$$(A + B + C - \pi) \text{ radians.}$$

PROP. 17. **The area of a spherical triangle is measured by $(A+B+C-\pi)a^2$, where A, B, C are the circular measures of its angles and a is the radius of the sphere.**

Let the opposite extremities of the diameters through the points A, B, C, be the points A′, B′, C′.

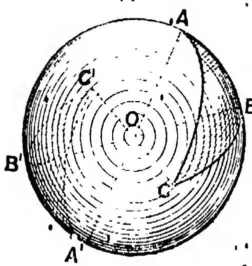


FIG. 91

If we add the lunes ABA′CA, BAB′CB, we obtain the hemisphere ABA′B′ plus the triangle ABC less the triangle CA′B′.

But the triangles CA′B′, C′AB can be divided into congruent portions, [Sch. (a), p. 97.

and therefore are of equal area.

If therefore we add the third lune CAC′BC we obtain an area equal to that of the hemisphere ABA′B′ plus twice the triangle ABC.

But the sum of the areas of the lunes is measured by $2(A+B+C)a^2$, and the area of the hemisphere by $2\pi a^2$.

The area of the triangle ABC is therefore equal to half their difference, that is to $(A+B+C-\pi)a^2$.

Examples XV.

1. Find, in square miles, the area of that portion of the Earth's surface which should be visible from a tower 200 feet high. [952 sq. miles.

2. What should be the distance apart of two ships when two men, each 40 feet above the level of the sea, can just see one another? [15·6 miles.

3. Show that the dip of the horizon and the extent of vision at sea from a height of x feet are approximately $1{\cdot}06\sqrt{x}$ minutes and $1{\cdot}23\sqrt{x}$ miles respectively.

4. If a solid sphere of radius a be viewed by an eye at a distance c from the centre, find the area of the visible portion. [$2\pi a^2 - 2\pi \frac{a^3}{c}$.

5. If the Moon's radius and distance be respectively 1000 and 240,000 miles, find (in miles) the distance of the edge of the visible portion from the boundary of that hemisphere of the sphere which is nearest to us. [$1\frac{1}{4}$ miles approx.

6. A sphere of radius r and centre O is cut by a right circular cone whose semi-vertical angle is α and vertex O. Show that the area on the sphere included within the cone is

$$2\pi r^2(1 - \cos \alpha)$$

7. A sphere is cut by two planes which meet in a tangent line to the surface, and enclose equal angles α, on opposite sides, with the diametral plane through the same line. Find the area of that portion of the spherical surface which is included between them. [$4\pi a^2 \sin \alpha$.

8. s_1, s_2 are the areas of the inner and outer surfaces of a spherical shell of thickness t, and s′ is the area of the spherical surface half-way between them. From Simpson's Rule or otherwise prove that the volume of the shell is

$$\tfrac{1}{6}t(s_1 + 4s' + s_2).$$

9. Prove that a plane bisecting a radius of a sphere at right angles divides the volume of the sphere into two parts in the ratio of 5 to 27.

10. A gasholder has a diameter of 30 feet, and the highest point of the roof, which is spherical, is 5 feet above the level of the top of the walls. Find how much the capacity exceeds what it would have been if the roof had been flat.

[1833 cubic feet.

11. A cone is made from a piece of metal whose form when developed is a circular quadrant. Show that its content bears to that of a sphere having the same radius as the quadrant the ratio of ·015.

12. If the volume of a uniform spherical shell be estimated by multiplying the thickness (t) into the area of the spherical surface which lies half-way between the inner and outer surfaces of the shell, will the result be too great or too small? [The error is twice the volume of the sphere whose diameter is t.]

13. A plane perpendicular to the base of a hemisphere, determined so as to cut off a portion such that the area of its curved surface is equal to half that of the base of the hemisphere, bisects a radius of the base at right angles.

14. Through any tangent line to a sphere of radius 6 cm., planes are drawn meeting it in circles of radii 3 and 5 cm. respectively; show that the area of the surface intercepted between them may be either 320 or 70 sq. cm. approximately.

15. A lens, plane on one side, spherical on the other, has a diameter of $1\frac{1}{2}$ in. and is $\frac{3}{8}$ in. thick; find the radius of the spherical surface and the volume of the lens.

[$\frac{15}{16}$ in.; ·36 cub. in

16. The Temperate Zones on the Earth lie between latitudes of $23\frac{1}{2}°$ and $66\frac{1}{2}°$; show that the areas called Arctic, Temperate, Tropical are roughly in the ratios 2 : 13 : 10.

17. Determine the semi-vertical angle of a spherical sector such that the area of its conical surface is n times that of the spherical portion. [$2 \cot^{-1} 2n$.

18. Determine a plane which shall cut a sphere so that the area of its section is equal to the difference between the areas of the two caps into which it divides the spherical surface. [Its distance from the centre is ·236 of the radius.

19. Two spheres of radii 13, 15 respectively intersect in a circle of radius 12, whose plane lies between the centres; prove that the ratio of the areas of each surface included within the other is 52 : 45.

20. A portion of a spherical surface is cut off between a movable plane and a fixed plane; prove that, if its area be constant, the movable plane touches a certain sphere.

21. If A is the pole of a spherical cap and P is any point on circumference of its base, the area is measured by $\pi \cdot AP^2$.

22. Cut a solid sphere by a plane so that the whole surfaces of the two portions may be in the ratio of 2 to 1.

[The heights of the segments have the ratio ·366.

23. The area and volume of a cylinder circumscribed to a sphere are respectively half as large again as the area and volume of the sphere.

24. In a given sphere determine a segment the volume of which is n times the volume of a cone on the same base and of the same height. The semi-vertical angle of the corresponding sector is $2 \tan^{-1}(2n-3)^{\frac{1}{2}}$.

25. A solid is formed by rotating a circular sector of angle 30° about one of its straight edges; compare its volume with that of the complete sphere. [·069.

26. The volume of a frustum of a sphere, of height h, the ends of which have radii r_1, r_2, is measured by

$$\tfrac{1}{6}\pi h(h^2+3r_1^2+3r_2^2).$$

27. The volume of the frustum of a sphere may be found by Simpson's Rule.

28. Estimate (by Simpson's Rule) the volume of a cask 3 feet high, the diameter of either end being 2 feet, and the diameter of the mid-section $2\frac{1}{2}$ feet. [13·0 cubic feet.

29. Estimate the volume of the cask in the preceding question by supposing it divided into two equal frustums of a right circular cone. [12·0 cubic feet.

30. If a concentric spherical hollow of two-thirds its radius is made within a given sphere, and parallel planes are drawn to touch the spherical hollow, the volume included between them is to that of the given sphere as 5 : 9.

31. If a quadrilateral be described on the surface of the earth, having three angles, right angles, and the two sides adjacent to them equal, and the radius of the earth be taken to be 4000 miles, the equal sides must be nearly 40 miles in length in order that the fourth angle may exceed a right angle by 20″.

*32. The base AB and the area of a spherical triangle ABC being given, the locus of the vertex C is a small circle which passes through A′, B′ the antipodal points of A, B. (*Cp.* Ex. 29, p. 104; the sum of the angles at A′, B′ can be shown to differ from the third angle of the triangle A′B′C by a constant amount.)

Lexell's Locus, 1781.

33. If the tangent cone drawn to a sphere, σ, from any point Q be denoted by τ, the areas on the surfaces σ, τ, lying between concentric spheres, centre O, are equal.

(Note that Prop. 13, p. 145, is a special case.)

34. From considerations of area show that any angle of a spherical triangle is greater than half the spherical excess, and hence that two sides of a triangle are together greater than the third.

ALTERNATIVE TREATMENT OF THE FUNDAMENTAL PROPOSITIONS *

2. Existence Postulates.

1. General Existence Postulate. *There exist in space an unlimited number of mutually intersecting surfaces called planes; in each plane an unlimited number of mutually intersecting lines called straight lines; in each straight line an unlimited number of points.*

Notation. Planes will be denoted by Greek letters, α, β, γ, . . . straight lines by small Roman letters, **a**, **b**, **c**, . . . and points by capital Roman letters, **A**, **B**, **C** . . .

2. Existence of Straight Lines Postulate. *Through two points* **A, B** *there exists one and only one straight line.*

Notation. The straight line containing two points **A, B** will be denoted by **AB** or **BA** indifferently. In the present chapter **AB** is taken to refer not only to the segment lying between the points, but also to all the points beyond **A** and beyond **B**.

† **Cor.** *Two straight lines which intersect do so in one point only.*

For if they met in two points, Post. 2 would be contradicted.

Scholium: Given any point **A** and a straight line **b** not containing **A**, there exist an unlimited number of straight lines containing **A** and some point in **b**.

For in **b** there exist an unlimited number of points **X, Y**, . . .
[Post. 1.

* Letters are chosen, except in the case of Props. 1 and 2, so that the figures drawn for the original treatment are still applicable.

† Only theorems which are in fact implicitly contained either in the enunciation or the proof of a proposition or postulate will be called Corollaries; other deductions not important enough to rank as propositions will be called Scholiums.

and given the pairs of points, **A** and **X**, **A** and **Y**, . . . there exist a corresponding number of straight lines **AX**, **AY**, . . . each containing **A** and some point in **b**. [Post. 2.

3. Existence of Planes Postulate. *Through any straight line* **a** *and a point* **B** *outside it there exists one and only one plane.*

Notation. The plane which contains a straight line **a** and an external point **B** is denoted by **aB**, and the plane which contains the straight line **AB** and the point **C** outside it is denoted by **AB,C**.

Def. Two straight lines are **parallel** if they

(i) lie in one plane,

(ii) have no common point, *i.e.* do not meet.

4. Existence of Parallel Straight Lines Postulate. *Through any point* **B** *outside a straight line* **a**, *there exists in the plane* **aB** *one and only one straight line which does not meet* **a**.

Cor. (i). *Through any point* **B** *outside a straight line* **a** *there exists one and only one straight line parallel to* **a**.

Cor. (ii). *Every straight line* **c** *lying in a plane* α *must meet at least one of any two given intersecting straight lines* **BA**, **BC** *lying in* α.

3. Intersection Postulates and Deductions.

Def. All the points common to two planes form their **join** or the **trace** of one plane upon the other.

* **5. Intersection of Two Planes Postulate.** *The join of two planes* α, β *which have a common point* **A** *is a straight line through* **A**.

Notation. The straight line which is the join of two planes α, β is denoted by $(\alpha\beta)$.

Cor. *Through two parallel straight lines* **a**, **b** *there exists one and only one plane.*

(For there is one plane by definition, and there can not be two, because two planes would then have two straight lines in common, which contradicts the postulate.)

Notation. The plane containing two parallel straight lines, **a**, **b** is denoted by **ab**.

* *Cp.* Euc. xi. 3.

The following three theorems are described as postulates because their proofs involve suppositions which seem absurd to a beginner unused to logical abstractions. Nevertheless, for the interest of more advanced students proofs are appended showing that they are implicitly contained in the preceding five postulates.

6. Intersection of Straight Line and Plane Postulate. *If a plane α has a point* **A** *in common with a straight line* **AB**, *it either contains it or has no other point in common with it.*

For suppose that α meets **AB** in a second point **B**, there exists outside **AB** and the plane α a point **X**. [Post. 1.

Through the straight line **AB** and the point **X** outside it there exists a plane **ABX**. [Post. 3.

The join of the two planes α, **ABX**, which have the points **A**, **B** in common, is a straight line containing **A**, **B**. [Post. 5.

But, given two points **A**, **B**, there is one and only one straight line which contains both, *i.e.* the straight line **AB**. [Post. 2.

That is, α either (i) has a second point in common with **AB**, and contains it, or (ii) has no other point in common with it.

Cor. *The whole of the straight line joining* **A**, **B** *two points of a plane* α *lies in* α.

Def. A straight line which has only one point in common with a plane is said to cut it.

7. Existence of Planes Postulate. *Through three points not in one straight line there exists one and only one plane.*

For through two points **A**, **B** there exists one and only one straight line **AB**. [Post. 2.

Through the line **AB** and the point **C** outside it there exists one and only one plane **AB**, **C**. [Post. 3.

Lastly, the plane **AB**, **C** must coincide with the planes **BC**, **A** and **CA**, **B**, because otherwise two planes would have in common three points not in one straight line, which contradicts Post. 5. [Post. 5.

Notation. The plane through three points **A**, **B**, **C** not in one straight line is denoted by **ABC**.

8. Existence of Planes Postulate. *Through two straight lines* **a**, **b** *meeting in* **A** *there exists one and only one plane.*

For in the straight line **b** there exists some point **X** other than **A**. [Post. 1.

This point is outside **a**, because otherwise, through **A**, **X**, there would be two straight lines **a**, **b**, which contradicts Post. 2. [Post. 2.

Through the straight line **a** and the point **X** outside it there is one and only one plane **aX** [Post. 3.

The whole of the straight line **b** joining **A**, **X**, two points of the plane **aX** lies within it [Post. 6, Cor.

That is, there is one plane, **aX**, containing **a**, **b**

There is only one, because if there were two planes containing **a**, **b**, then, through the straight line **a** and the point **X** outside it, there would exist two planes, which contradicts Post. 3. [Post 3.

Notation. The plane containing two straight lines **a**, **b** meeting at **A** is denoted by **ab**.

4. The Relations between a Plane and a Straight Line.

Def. A straight line and a plane are **parallel** if they have no common point.

Prop. 1. **If a straight line a is parallel to any straight line b within a plane β, it is either parallel to β or contained by it.**

For **a**, **b** being parallel are contained by a plane **ab**. [Def.

The plane **ab** is either different from the plane β or is identical with it.

In the first case, the join of the two planes **ab**, β is their common line **b**. [Post. 5.

Therefore any point common to **a** (which lies in **ab**) and β lies in **b**.

But **a**, **b** have no point in common, [Def.

therefore **a**, β have no point in common, and are parallel. [Def.

In the second case, **a** is contained in β.

Scholium: Conversely, if a straight line **a** is parallel to a plane β it is parallel to some line through each point in β.

In the plane β consider any point **X**

Through the straight line **a** and the point **X** outside it there exists a plane **aX**. [Post. 3.

The join of the planes **aX**, β, having the common point **X**, is some straight line **XY**. [Post. 5.

Since the straight line **a** and the plane β have no point common (Def.), the straight line **a** and the straight line **XY** (lying in β) have no point in common **X**, and (being in the same plane **aX**) they are therefore parallel. [Def.

Scholium (a): Any plane α which passes through a point **A** outside a line **b** and is parallel to **b** contains the line through **A** parallel to **b**.

For, as in the last scholium, the join of the planes α, **bA**, is a straight line through **A** parallel to **b**. [Posts. 3, 5.

Scholium (b): Through any point **A** outside a plane β there exists an unlimited number of straight lines parallel to β.

For in the plane β there exist an unlimited number of mutually intersecting straight lines. [Post. 1.

Through **A**, a point outside each of these straight lines, there exists one and only one straight line parallel to each one of them. [Post. 1, Cor. (i).

These lines are all distinct, because otherwise two intersecting straight lines in β would be parallel to the same straight line through **A**, which contradicts Post. 1. [Post. 1, Cor. (i).

These lines through **A** being all parallel to straight lines in β are either parallel to β or contained in β. [Prop. 1.

Finally, **A** is outside β, and therefore these lines can not be contained in β, they are therefore all parallel to β. [Def.

The possible relations between a straight line **a** and a plane β may now be stated :—

The plane β may (i) contain **a**,
(ii) cut **a** in one point,
(iii) be parallel to **a**.

5. The Relations between Two Straight Lines.

Def. Two straight lines **a, b** are co-planar if some plane α exists which contains both **a** and **b**.

Two straight lines **a, b** are skew if they are not co-planar; that is, if no plane exists which contains both.

Cor. (i) to Post. 5. If two straight lines **a, b** are co-planar, no plane can contain **a** and cut **b** in a point **C** outside **a**.

For if such a plane α existed, through the straight line **a** and the point **C** outside it there would exist two planes α and **aC**, which contradicts Post. 5.

Cor. (ii) to Post. 5. If any plane α contains a straight line **a** and cuts another line **b** in a point **C** external to **a**, **a, b** are skew.

For if a plane β existed containing **a, b**, then, through the straight line **a** and the point **C** outside it, there would exist two planes α, β, which contradicts Post. 5. [Post. 5.

Cor. (iii) to Post. 5. If a plane α cuts a straight line **b** in **B**, all straight lines in α, except those through **B**, are skew with respect to **b**.

For if there be any straight line **a** in α not passing through **B** and not skew with respect to **b**, there would exist a plane containing **a**, **b**, and through the straight line **a** and the point **B** outside it there would exist two planes α, **ab**, which contradicts Post. 5. [Post. 5.

The possible relations between two straight lines may now be summed up as follows:—

Two straight lines are either

<table>
<tr><td>Intersecting</td><td colspan="2">Non-intersecting</td></tr>
<tr><td>Intersecting</td><td>Parallel</td><td>Skew</td></tr>
<tr><td colspan="2">Co-planar</td><td>Skew</td></tr>
</table>

6. The Relations between Two Planes.

Def. If two planes have no common point they are parallel.

Prop. 2. **Through any point A outside a plane β there exists at least one plane parallel to β.** *

In the plane β there exist two intersecting straight lines **XY**, **XZ**. [Post. 1.

Through the point **A**, outside the straight lines **XY**, **XZ**, there exist straight lines **AB**, **AC** respectively parallel to **XY**, **XZ**. [Post. 4, Cor. (i).

The straight lines **AB**, **AC**, parallel respectively to the straight lines **XY**, **XZ** within the plane β, are parallel to β or contained by it. [Prop. 1

* That there is *only* one is proved as a corollary to Prop. 4.

Since the point A is outside β, AB, AC are not contained in β, and are therefore parallel to β.

Through the straight lines AB, AC meeting in A there exists a plane ABC; it remains to be proved that β, ABC are parallel. [Post. 8.

Since AB, AC are parallel to the plane β, no straight line in β meets AB or AC. [Def.

Also, every straight line in the plane ABC meets one at least of the two intersecting lines AB, AC within it. [Post. 4, Cor. (ii).

Therefore the planes β, ABC have no straight line in common. Hence the planes β, ABC have no common point, for their join, if they have a common point, is a straight line. [Post. 5.

That is, the planes β, ABC are parallel.

Cor. (i). *If two intersecting straight lines* AB, AC *are each parallel to a plane β, the plane containing them,* ABC, *is parallel to β.*

* **Cor.** (ii). *If two intersecting straight lines* AB, AC *are respectively parallel to two intersecting straight lines* XY, YZ, *the plane* ABC *containing the first pair is parallel to* XYZ, *that containing the second pair.*

The relations between two planes may now be summed up.

Two planes may either (i) cut in a straight line,
(ii) be parallel.

7. Three Planes.

Prop. 3 (a). **The joins of three planes α, β, γ, no two of which are parallel, are coincident, concurrent, or mutually parallel straight lines.**

† (*b*) **If two planes α, β are parallel, and neither is parallel to a third plane γ, their joins with γ are parallel.**

(*a*) For the three planes either

(i) have two points A, B or more common to all,
(ii) have only one point A common to all,
or (iii) have no point common to all.

* Euc. xi. 15. † Euc. xi. 16.

In the first case, the straight line **AB** joining the points **A, B** lying in the three planes α, β, γ is also common to all. [Post. 6, Cor.

In the second case, the joins of the planes α and β, β and γ, γ and α, which all have the common point **A**, are three straight lines concurrent in **A**. [Post. 5.

In the third case, the join of the planes α, β, which have a common point (not being parallel), is a straight line $(\alpha\beta)$, and similarly the joins of β, γ and γ, α are $(\beta\gamma)$ and $(\gamma\alpha)$. [Post. 5.

But $(\alpha\beta)$ and $(\beta\gamma)$ are in the same plane β, and have no common point, because otherwise that point would be common to all three planes α, β, γ. They are therefore parallel. [Def.

Similarly $(\beta\gamma)$ and $(\gamma\alpha)$ are parallel, and also $(\gamma\alpha)$ and $(\alpha\beta)$; that is, the three joins are mutually parallel.

(*b*) If α, β are parallel, the joins $(\alpha\gamma)$ and $(\beta\gamma)$ can not meet, for if they did these would be a point common to α, β, and γ, which contradicts the hypothesis that α, β have no common point. [Def.

But $(\alpha\gamma)$ and $(\beta\gamma)$ lying in the same plane γ and having no common point are therefore parallel. [Def.

PROP. 4. **If two planes α, β are each parallel to a third γ, then α, β are also parallel.**

For suppose that α, β have some common point **A**.
In the plane γ there exists a point **Z**, [Post. 1.

and in the plane β there exists a point **Y** outside the plane α and the straight line **AZ**. [Post. 1.

Through the three points **AYZ** not in one straight line there exists a plane **AYZ**. [Post. 7.

Since the two parallel planes α, γ have respectively common points **A**, **Z** with the plane **AYZ**, their joins with it are two parallel straight lines **AX, ZW** (say). [Prop. 3 (*b*).

Since the two parallel planes β, γ have respectively common points **Y, Z** with the plane **AYZ**, their joins with it are two parallel straight lines **AY, ZW**. [Prop. 3 (*b*).

The two straight lines **AX**, **AY** are distinct, because the point **Y** was chosen outside the plane α, and is therefore outside **AX**, which is contained in α.

Therefore, through the point **A**, outside the straight line **ZW**, there exist two straight lines **AX**, **AY**, both parallel to **ZW**, which contradicts Post. 4, Cor. (i). [Post. 4, Cor. (i).

But the only **assumption** that has been made is the initial one that α, β have a common point **A**; that is now proved impossible, and α, β are therefore parallel. [Def.

Cor. *If a plane α cuts one of two parallel planes β, γ, it cuts the other also,* i.e. *through any point there is only one plane parallel to a given plane.*

For suppose α cuts β, then if α did not cut γ the planes α, β would be parallel to γ, and therefore parallel to one another, which contradicts the original hypothesis.

The possible relations between three planes may now be summed up as follows:—

Planes	Joins
No two parallel	Coincident, three concurrent or three parallel
Two parallel	Two parallel
Three mutually parallel	None

8. Two Planes and a Straight Line.

Prop. 5. If two planes α, β are parallel, any straight line l which cuts α in some point A cuts β also.

There exists a point **B** in the plane β outside the straight line **l**. [Post. 1.

Through the straight line **l** and the point **B** outside it there exists a plane **lB**. [Post. 3.

Since the two planes α, β are parallel, and they have respectively points **A**, **B** common with the plane **lB**, their joins with **lB** are two parallel straight lines **AX**, **BY** (say). [Prop. 3 (*b*).

The point **A**, lying in the plane α which has no point common with β, is outside any straight line **BY** lying in β.

Through the point **A**, outside the straight line **BY**, there exists in the plane **lB** one and only one straight line, **AX**, which does not meet **BY**. [Post. 4.

That is, the straight line **l** meets **BY**, and therefore cuts the plane β also.

Cor. *If two planes α, β are parallel, a straight line* **l** *which is parallel to or contained by β is parallel to or contained by α.*

For if this were not the case Prop. 5 would be contradicted.

Scholium (*a*): If a straight line **l** is contained by α, one of two intersecting planes α, β, and is parallel to β, it is parallel to the join ($\alpha\beta$). [Defs.

Scholium (*b*): If a straight line **l** is parallel to each of two intersecting planes α, β, it is parallel to their join (α, β).

[Use Sch., p. 160, or Sch. (*a*), above.]

9. One Plane and Two Straight Lines.

PROP. 6. **If two straight lines l, m are parallel, any plane** α **which cuts l in some point A cuts m also.**

Through the parallel straight lines **l**, **m** there exists a plane **lm**. [Post. 5, Cor.

The join of the two planes α, **lm** having the common point **A** is the straight line **AX** (say). [Post. 5.

Through the point **A**, outside the straight line **m**, there exists in the plane **lm** only one straight line **l**, which does not meet **m**. [Post. 4.

That is, AX, lying in the plane lm, meets m in some point Y, which is a point common to m and a.

a can not contain m, because then through the straight line m and the point A outside it there would be the two planes lm and a, which contradicts Post. 3. [Post. 3.

That is, a cuts m. [Post. 6.

Cor. *If two straight lines* l, m *are parallel, a plane a which is parallel to or contains* m *is also parallel to or contains* m.

For if this were not the case Prop. 6 would be contradicted.

10. Three Straight Lines.

Prop. 7. **If two straight lines l, n are each parallel to a third, m, they are parallel to each other.**

(i) l, n can not meet in any point A, for, if they did, through the point A, outside the straight line m, there would be two straight lines l, n parallel to m, which contradicts [Post. 4, Cor. (i).

(ii) l, n must be proved co-planar.

There exists in n a point B outside l. [Post. 1.

Through the line l and the point B outside it there exists a plane lB. [Post. 3.

Also through the pairs of parallel straight lines l and m, m and n, there exist planes lm, mn. [Def.

The joins of these three planes lB, lm, mn, no two of which are parallel, are coincident, concurrent, or parallel. [Prop. 3 (a).

They are not coincident or concurrent because two joins, l, m, are parallel. [Hyp.

That is, they are all parallel, and the join of lB and mn is therefore parallel to m. [Prop. 6.

But through the point B outside the line m there exists only one straight line parallel to it, viz. n. [Post. 4, Cor. (i).

Which is therefore the join of lB and mn.

That is, l, n are co-planar, and since they do not meet they are parallel. [Def.

Scholium: Three straight lines **l, m, n**, such that each cuts the other two, are concurrent or co-planar.

For if **l, m** meet in **A**,

n must either (i) pass through **A**, in which case the lines are concurrent,

or (ii) meet **l, m** in points **B, C**.

In the second case, through the two intersecting straight lines **l, m** there exists a plane **lm**. [Post. 8.

And the whole of the straight line **n** containing **B, C**, two points of the plane **l, m** is contained in **lm**. [Post. 6, Cor.

That is, **l, m, n** are co-planar.

The reasoning may clearly be extended to cover the case of any number of straight lines such that each cuts all the others.

§15. Propositions to replace Prop. 12, p. 26.

* Prop. 12 (a). **Through any given point A there exists one and only one straight line normal to a given plane** α.

Case (i) When **A** lies outside the plane α.

In the plane α there exists a straight line **l** outside **A**. [Post. 1, p. 157.

Through the straight line **l** and the point **A** outside it there exists a plane **lA**. [Post. 3, p. 158.

Through **A** in the plane **lA** there exists a straight line **AB** meeting **l** at right angles in some point **B**. [Euc. i. 12.

Through **B** in the plane α there exists a straight line **BC** meeting **l** at right angles in **B**. [Euc. i. 11.

Through the straight lines **AB, BC** meeting at **B**, there exists one and only one plane **ABC**. [Post. 8, p. 159.

Through **A** in the plane **ABC** there exists a straight line **AP**, meeting **BC** at right angles in some point **P**. [Euc. i. 12.

* Euc. xi. 11, 12, 13, which cover part of the ground only.

(It is now required to prove that **AP** is normal to the plane α, and this is the same thing as proving that it is perpendicular to two intersecting straight lines such as **l** and **BC**.)

The line **l** perpendicular to the two intersecting straight lines **BA, BC** is normal to the plane **ABC** containing them, [Prop 11, p. 24.

l is therefore perpendicular to all straight lines in the plane **ABC**, and in particular to **AP**, [Def 3, p. 24.

and **AP, BC** are perpendicular by hypothesis.

Therefore **AP** being perpendicular to the two intersecting straight lines **l, BC** is normal to the plane α [Prop. 11, p. 24.

No other straight line through **A, AQ**, is normal to α, because if it were, through **A** there would be two straight lines **AP, AQ** both perpendicular to the straight line **PQ** (lying in the plane α), [Def. Post 6, Cor.

and therefore a triangle **APQ** with two angles right angles, which contradicts [Euc. i. 32.

Case (ii) in which **A** lies within α.

There exists a plane β intersecting α in some straight line **XY**. [Posts. 1, 5, p. 157.

Through **A** there exists in the plane α a straight line **AB** meeting **XY** at right angles in some point **P**. [Euc. i. 11 or 12.

(It makes no difference to the proof whether **P** coincides with **A** or not.)

Through **P** there exists in the plane β a straight line **PQ** perpendicular to **XY**. [Euc. i. 11.

Through **AB, PQ** meeting at **P** there exists a plane **ABQ**. [Post 8, p. 159.

Through A there exists in the plane **ABQ** a straight line **AZ** perpendicular to **AB**. [Euc. i. 11.

(It is now required to prove that **AZ** is normal to α.)

XY being, by hypothesis, perpendicular to the intersecting lines **AB, PQ** is normal to the plane **ABQ** containing them. [Prop. 11, p. 24.

XY is therefore perpendicular to the straight line **AZ** contained in the plane **ABQ**. [Def. 3, p. 24.

AZ is also perpendicular to **AB**. [Hyp.

Therefore, **AZ** being perpendicular to intersecting lines **XY, AB** is normal to the plane a which contains them. [Prop. 11, p. 24.

No other straight line through **A**, **AW**, is normal to a, because in that case there would exist containing the two straight lines **AZ, AW** a plane **AZW**. [Post. 8, p. 159.

This plane having the point **A** in common with a would cut a in some straight line **AT**, [Post. 5, p. 4.

and since **AT** lies in a, the straight lines **AZ, AW**, both normal to a, would be perpendicular to **AT**. [Def. 3, p. 24.

That is, through a point in a straight line **AT** there would be two straight lines perpendicular to it, lying in the same plane with it, which contradicts the Euclidean axiom that all right angles are equal.

The assumption that there exists a second normal through **A** being disproved, it follows that through **A** there exists one and only one normal to a, viz. **AZ**.

Prop. 12 (*b*). **Through any given point A there exists one and only one plane normal to a given straight line l.**

If **A** lies outside **l**, there exists through **A** a straight line **AX** parallel to **l**. [Post. 4, Cor. (i) p. 158.

If **A** lies within **l**, we may refer to **l** itself as the line **AX**.

There exist through **AX** and two points **P, Q** outside it two planes **AXP, AXQ**. [Posts. 1, 3, p. 157.

Through **A** there exist in the planes **AXP, AXQ** respectively straight lines **AY, AZ** perpendicular to **AX**. [Euc. i. 11.

Through the straight lines **AY, AZ** there exists a plane **AYZ**. [Post. 8, p. 159.

The line l, which either coincides with AX or is parallel to it, is perpendicular to AY, AZ in either case, [Defs. 1, 2, p. 24.

and therefore is normal to the plane AYZ, which contains both. [Prop. 11, p. 24.

Through A no other plane ALM can exist normal to l, because if there were such a plane it would contain two intersecting straight lines AL, AM, [Post. 1, p. 157.

to which l would be perpendicular. [Def. 3, p. 24.

AX, being parallel to l or coinciding with it, must also be perpendicular to these lines; [Defs. 1, 2, p. 24.

and therefore would be normal to the plane ALM containing them; [Prop. 11, p. 24.

through the straight line AX and the point L there would exist a plane AXL; [Post. 3, p. 158.

which, having the point A in common with the plane AYZ, would meet it in a straight line AT; [Post. 5, p. 158.

and finally, through A there would exist in the plane AXL two straight lines AL, AT both perpendicular to AX, [Def. 3, p. 24.

which contradicts the Euclidian axiom that all right angles are equal.

That is, through any point A there is one and only one plane AXY normal to the straight line l.

Proposition 16, p. 34, is of no great importance, and may well be omitted when time is lacking.

REFERENCES TO THEOREMS IN PLANE GEOMETRY

Euc. i. 4.—Two triangles are congruent if two sides and the included angle of the one are congruent, respectively, with two sides and the included angle of the other.

Euc. i. 5.—In an isosceles triangle the angles opposite the congruent sides are congruent.

Euc. i. 6.—If two angles of a triangle are congruent, the sides opposite the equal angles are congruent.

Euc. i. 8.—Two triangles are congruent if the three sides of the one are congruent, respectively, to the three sides of the other.

Euc. i. 11.—To draw a straight line at right angles to a given straight line from a given point in the same.

Euc. i. 12.—To draw a straight line perpendicular to a given straight line of unlimited length from a given point without it.

Euc. i. 16.—An exterior angle of a triangle is greater than either of the two opposite interior angles.

Euc. i. 18.—If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.

Euc. i. 19.—If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.

Euc. i. 20.—Two sides of a triangle are together greater than the third.

Euc. i. 25.—If two sides of a triangle are congruent, respectively, with two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.

Euc. i. 26.—Two triangles are congruent if two angles and the included side of the one are congruent, respectively, with two angles and the included side of the other.

Euc. i. 29.—If two parallel lines are cut by a transversal, the exterior angle is equal to either the interior or opposite angles.

Euc. i. 32.—The sum of the three angles of a triangle is two right angles.

Euc. i. 33.—If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.

Euc. i. 34.—The opposite sides of a parallelogram are congruent.

Euc. vi. 2.—If a line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionately.

Euc. vi. 3.—The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

Euc. vi. 4.—Two mutually equiangular triangles are similar.

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